Laplacian spectra of, and random walks on, complex networks: Are scale-free architectures really important?

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We study the Laplacian operator of an uncorrelated random network and, as an application, consider hopping processes (diffusion, random walks, signal propagation, etc.) on networks. We develop a strict approach to these problems. We derive an exact closed set of integral equations, which provide the averages of the Laplacian operator's resolvent. This enables us to describe the propagation of a signal and random walks on the network. We show that the determining parameter in this problem is the minimum degree q_m of vertices in the network and that the high-degree part of the degree distribution is not that essential. The position of the lower edge of the Laplacian spectrum λ_c appears to be the same as in the regular Bethe lattice with the coordination number q_m. Namely, $\lambda_c > 0$ if $q_m > 2$, and $\lambda_c = 0$ if $q_m \leq 2$. In both of these cases the density of eigenvalues $\rho(\lambda) \rightarrow 0$ as $\lambda \rightarrow \lambda_c + 0$, but the limiting behaviors near λ_c are very different. In terms of a distance from a starting vertex, the hopping propagator is a steady moving Gaussian, broadening with time. This picture qualitatively coincides with that for a regular Bethe lattice. Our analytical results include the spectral density $\rho(\lambda)$ near λ_c and the long-time asymptotics of the autocorrelator and the propagator.

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I. INTRODUCTION

The Laplacian spectra of random networks determine a wide circle of processes taking place on these networks, see, e.g., [1–10] and references therein. Random walks, signal propagation, synchronization, and many others are among these processes. This is why the problem of Laplacian spectra of random networks (especially, of its low-eigenvalue part which determines the long-time behavior of relevant processes) is considered as one of the central problems of graph theory and the science of complex networks. In this paper we essentially resolve this problem applying the strict statistical mechanics approach to sparse uncorrelated random networks with arbitrary degree distributions. These random graphs constitute a basic class of complex networks.

One should note that leading contributions to the spectra and the asymptotics of the random walk autocorrelator were found by Bray and Rodgers in 1988 in the particular case of the Erdős-Rényi graphs [3]. It is important that these classical graphs necessarily have dead ends and vertices with two connections. We will show that the absence of these vertices in a network qualitatively changes the spectra and the random walk asymptotics. Random walks on hierarchically organized, deterministic, scale-free graphs were studied by Noh and Rieger in Ref. [11]. Due to a very specific organization of these graphs, their results are not applicable to equilibrium networks. This is also the case in respect of the recent numerical work of Kujawski, Tadić, and Rodgers [2], who found the autocorrelator of a random walk on a grown scale-free network by performing extensive numerical simulations. Their network was strongly correlated in contrast to

the configuration model of a random graph, which we use in this work. For an arbitrary network, Noh and Rieger in Ref. [12] found a mean first passage time and an average return time, which are rather integrated characteristics. (According to the Kac formula [13], the mean return time for a vertex coincides with the inverse probability to find the particle at this vertex in the final equilibrium state of the process, after complete relaxation.) Kim and Kahng, Ref. [8], described the Laplacian spectrum of the static model of an uncorrelated complex network in the more simple case of high density of connections (high mean degrees).

For the sake of clearness, let us remind the reader of basic notions and terms for random networks. For more detail see [14–20]. A graph is completely defined by its $N \times N$ adjacency matrix $\hat{A}$, whose elements A_{ij} are the numbers of edges between i and j. The vertex degree of vertex i is the number of edges, attached to this vertex, $q_i = \Sigma_{j=1}^N A_{ij} = \Sigma_{j=1}^N A_{ji}$. In random networks, q_i is a random variable with a degree distribution $\Pi(q) = \langle \delta(q - q_i) \rangle$.

In traditional mathematical models, $\Pi(q)$ is a rapidly decaying function with a well-defined scale. For example, in the Erdős-Rényi model [21], which is a standard one, $\Pi(q)$ is a Poisson distribution decaying as $(\bar{q}/eq)^q$, i.e., faster than any exponent. In contrast to these models, in most of the real-world networks degree distributions are heavy tailed. After the work [22], they are usually approximated by a power law $\sim q^{-\gamma}$ in the range of sufficiently high degrees. Note that the validity of this fitting is limited because real-world networks are small (even the WWW has only about 10^{10} vertices), and so high degrees are not observable. It is commonly believed that the "scale-free networks" are greatly distinguished from the others in every aspect. This widespread belief actually implies a division of all networks into two classes: "scale-free networks" and all others. In contrast to these beliefs, we here show that scale-free (or, more generally, heavy tailed) architectures of networks are not essen-

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TABLE I. Asymptotics of the Laplacian spectral density $\rho(\lambda)$, autocorrelator $\bar{P}_0(t)$ and propagator $\bar{P}_l(t)$ for the random uncorrelated networks where $\Pi(q_m)$ is essentially distinct from 0 and 1. Here $p_l^{(eq)} = P_l(t \to \infty)$ are stationary values of the correlator given by Eq. (38) for $l=0$ and Eq. (45) otherwise, $\beta = \pi(q_m -1)^{1/4} \ln(q_m-1)$. The values of the parameters in the preexponential factors are $\xi = 9/10$ and $4/3$, $\eta = -7/30$ and $1/18$, and $\zeta = 13/30$ and $5/18$ for $q_m=1$ and 2, respectively. v and D are determined by the full form of the degree distribution $\Pi(q)$.

	Minimum vertex degree $q_m > 2$	Minimum vertex degree $q_m=1$ or 2
Spectral edge λ_c	$q_m - 2\sqrt{q_m-1}$	0
Spectral density	$\exp[\beta/2\sqrt{\lambda-\lambda_c} - d\exp(\beta/\sqrt{\lambda-\lambda_c})]$, Eq. (55)	$p_0^{(eq)}\delta(\lambda) + \text{const } \lambda^{-\xi}\exp(-a/\sqrt{\lambda})$, Eqs. (64) and (76)
Autocorrelator	$\exp(-\lambda_c t - \beta^2 t/\ln^2 t)$, Eq. (56)	$p_0^{(eq)} + \text{const } t^\eta \exp[-3(a/2)^{2/3}t^{1/3}]$, Eqs. (65) and (77)
Propagator at $l \sim t$	$1/\sqrt{2\pi Dt}\exp[-(l-vt)^2/2Dt]$, Eq. (62)	$1/\sqrt{2\pi Dt}\exp[-(l-vt)^2/2Dt]$, Eqs. (62) and (79)
Propagator at $t \to \infty$	$\mu_0^l(-\lambda_c)\exp(-\lambda_c t - \beta^2 t/\ln^2 t)$	$p_l^{(eq)} + ct^{-\zeta}\exp[-3(a/2)^{2/3}(t-l/v)^{1/3}]$, Eqs. (70) and (80)

tial for a lower edge of the Laplacian spectra and the long-time behavior of random walks characteristics. The resulting dependences are determined by the minimum degree of vertices in a network. Heavy tails determine some coefficients and amplitudes but not a type of these singularities.

In this paper we study properties of the Laplacian operator

$$L_{ij} = q_i\delta_{ij} - A_{ij} \qquad (1)$$

on an uncorrelated random network near the lower edge of its spectrum, and, respectively, the hopping motion of some carrier ("signal") from one vertex to another at large times. This operator corresponds to the process described by the following dynamic equations for the probability $p_{ij}(t)$ that at time t a particle is at vertex i if at time 0 it was at vertex j,

$$\dot{p}_{ij}(t) = \sum_{k=1}^{N} A_{ik}p_{kj}(t) - q_i p_{ij}(t), \quad p_{ij}(0) = \delta_{ij}. \qquad (2)$$

This is a random walk where the rate of hopping along any edge is set to one. Other versions of the Laplace operator and corresponding processes, which are also widely discussed in literature, are listed in Appendix A.

We use the configuration model of an uncorrelated network [23,24], which is a maximally random network with a given degree distribution. It is convenient that (i) this model is statistically homogeneous, (ii) all its vertices are statistically independent, and (iii) it has a locally treelike structure. We consider only infinite networks, that is, first we tend the total number of vertices N to infinity (the thermodynamic limit) and only afterwards study network characteristics. If, say, we study a random walk, then a particle should be still much closer to an initial vertex than the diameter of the network $\sim\ln N$. In other words, we consider the process at so short times that the number of vertices, where the walking particle may be found, is negligible compared with the network's size N. We will see that this imposes strong limita-

tions to the applicability of our results due to the "small world" feature of the networks under consideration.

We will show that for the Laplacian spectra and for random walks, the crucial property of the random uncorrelated network is the minimum degree q_m of its vertices. We suppose that the value of the degree distribution at q_m essentially differs from 0 and 1. We also assume that $q_m > 0$, because the contribution of isolated vertices is trivial. Our results are summarized in Table I and in Fig. 1. Note an unusual singularity of the spectral density in the case $q_m > 2$.

As is natural, the calculation of the spectrum is reduced to the study of the trace of the Laplace operator's resolvent. To describe the propagation of the signal in the network, one must know the nondiagonal elements of the resolvent. Here we calculate the asymptotics of their average values. It allows us to obtain the time and distance dependences of the signal's propagator $\bar{p}_{ij}(t) = \bar{p}_l(t)$ when the distance between
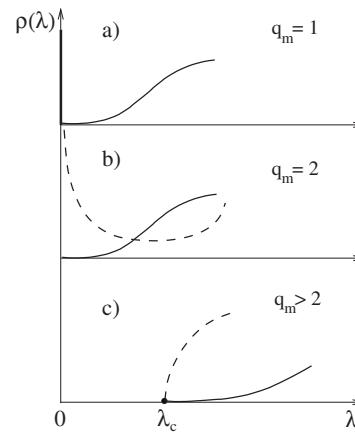

FIG. 1. Laplacian spectral density $\rho(\lambda)$ for networks with different minimum vertex degree q_m (solid lines): (a) $q_m=1$, (b) $q_m=2$, and (c) $q_m>2$. The dashed lines show $\rho(\lambda)$ for an infinite chain (b) and for the Bethe lattice with the coordination number $q_m>2$ (c). The vertical line at $\lambda=0$ represents a δ-function peak.

initial i and final j vertices, l, is much smaller than the diameter of the network, $\bar{l} \sim \ln N$.

Why is the minimum vertex degree so important in these problems? Note that in respect of random walks and Laplacian operator related problems, infinite uncorrelated networks are equivalent to infinite Bethe lattices with coinciding degree distributions. (Recall that a Bethe lattice is an infinite tree without borders.) Let us compare two Bethe lattices—random, with the minimum coordination number q_m, and regular, with the coordination number equal to q_m. It is clear that the autocorrelator in the random Bethe lattice cannot decay slower than in the regular Bethe lattice with this q_m. If $q_m > 2$, then in this regular Bethe lattice, $\bar{p}_{ii}(t) = \bar{p}_0(t) \sim t^{-3/2} \exp(-\lambda_c t)$, where

$$\lambda_c = q_m - 2\sqrt{q_m - 1}. \tag{3}$$

λ_c is also the spectral boundary in the Laplacian eigenvalue density $\rho(\lambda)$ of this regular Bethe lattice, where $\rho(\lambda) \sim \sqrt{\lambda - \lambda_c}$, near λ_c. (For the calculation of the autocorrelator and Laplacian spectrum of a regular Bethe lattice see, e.g., Refs. [25–27]. The work [27] describes details of these calculations based on the application of the generating function approach to Eq. (2) for a regular, homogeneous case.) Thus, the spectral boundary for an infinite uncorrelated network in principle cannot be lower than that for the regular Bethe lattice with the same q_m. Moreover, these borders coincide. The reason for this is the following feature of the configuration model of an uncorrelated network. Let the number of vertices N in this model approach infinity. Then the mean number of given finite regular subgraphs with coordination number q_m grows proportionally to N. We stress that although this number rapidly decreases with a size of these subgraphs, it is proportional to N for any given subgraph size. In the arbitrarily large subgraphs, the lowest eigenvalues are arbitrarily close to the spectral boundary of the corresponding regular Bethe lattice. The number of these eigenvalues is proportional to the number of these subgraphs and so proportional to N. Now recall that the total number of eigenvalues in the spectrum is N. Therefore, indeed, the spectral borders for the configuration model and for the regular Bethe lattice with q_m coincide.

The statistics of these regular tree subgraphs determine the singularity of the resulting $\rho(\lambda)$ at the edge λ_c. The rapid decrease of the number of these subgraphs with their size results in specific singularities, with all derivatives zero, represented in Table I.

The random networks with $q_m = 1, 2$ markedly differ from those with $q_m > 2$. In the configuration model with $q_m = 1, 2$, chains and chainlike subgraphs are statistically essential. Let us first discuss the case $q_m = 2$. The Bethe lattice with coordination number 2 is a usual infinite chain. It has the spectral boundary $\lambda_c = 0$. Near this edge, $\rho(\lambda) \sim \lambda^{-1/2}$. Thus, the edge of the spectrum of the uncorrelated network with $q_m = 2$ is zero. We will show that the statistics of chain subgraphs in this configuration model differ from those for the case $q_m > 2$. This results in different asymptotics presented in Table I and, schematically, in Fig. 1.
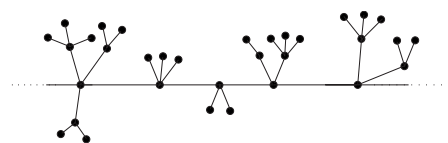

FIG. 2. Chain with finite treelike branches in a random network with the minimum vertex degree $q_m = 1$.

If $q_m = 1$, chainlike subgraphs are also present in the configuration model. There are, however, more chains (see Fig. 2) with branches attached. Nonetheless, these subgraphs result in the spectrum edge $\lambda_c = 0$ and in the same asymptotics as for $q_m = 2$. When $q_m = 1$, numerous finite components are present in the network. Their mean number is proportional to N. Each of the connected components gives one zero eigenvalue in the spectrum. This leads to a δ-function peak at $\lambda = 0$ in the spectral density.

The obtained singularities of $\rho(\lambda)$, with all derivatives zero, have a direct consequence for observations in finite networks. The approach of $\lambda_2(N)$, the minimal nonzero eigenvalue, to λ_c is extremely slow. Even in a huge uncorrelated network, the observed minimum eigenvalue λ_2 will be far from the spectral edge λ_c predicted for an infinite network. Note that a very slow convergence of λ_2 was recently observed in the numerical work of Kim and Motter [28], in which λ_2 and λ_c were compared for networks up to 4000 vertices. One should also note that the obtained asymptotics of the autocorrelator can be observed only in extremely large networks. We will indicate the range of network sizes, where our formulas are valid.

In Sec. II we strictly formulate the problem. In Sec. III we derive a basic set of integral equations. Solving these equations enables us to obtain the Laplacian spectrum $\rho(\lambda)$ for uncorrelated random networks and to describe the random walk on the networks in the thermodynamic limit. In Sec. IV we study the final value of the propagator $\bar{p}_i^{(\mathrm{eq})} = \bar{p}_l(t \to \infty)$, which is the equilibrium probability to find a signal at distance l from a starting vertex. We describe $\bar{p}_i^{(\mathrm{eq})}$ in terms of l and of the degree distribution $\Pi(q)$. Furthermore, we find the coefficient of the $\delta(\lambda)$ term in the Laplacian spectrum. In Sec. V we present general solutions of the integral equations of Sec. III and analyze them in three distinct cases, $q_m > 2$, $q_m = 2$, and $q_m = 1$. In Sec. VI we summarize our results and methods and discuss conditions for their applicability. Technical details are given in the Appendixes. In Appendix A we describe various forms of the Laplacian and corresponding processes. Appendix B contains necessary information about the Z-transformation (generating functions) in application to the statistics of connected components in the configuration model. In Appendix C we derive our basic relations: (i) Eq. (19) and equivalent Eq. (C7) and (ii) expression (20) of the autocorrelator in terms of the solution of Eq. (19) or Eq. (C7). In Appendix D we derive linear recursion relation (23). In Appendix E we find the asymptotics of the solution of Eq. (C7) in all three cases, $q_m > 2$, $q_m = 2$, and $q_m = 1$. In Appendix F we obtain the lowest eigenvalue for recursion (23) and the corresponding eigenfunction.

II. FORMULATION OF THE PROBLEM

The problem of the Laplacian spectrum of a random network is completely equivalent to that of the time dependence

of the averaged autocorrelator $\bar{p}_0(t) = \langle p_{ii}(t) \rangle$ for a random walk. This autocorrelator is the probability that a particle returns to the starting vertex after a time t. This quantity is related to the eigenvalue density

$$\rho(\lambda) = \frac{1}{N} \left\langle \sum_{n=1}^{N} \delta(\lambda - \lambda_n) \right\rangle \qquad (4)$$

in the following way:

$$\bar{p}_0(t) = \int_0^\infty d\lambda e^{-\lambda t} \rho(\lambda), \qquad (5)$$

where λ_k are (non-negative) eigenvalues of the Laplace operator on the network

$$\hat{L}\mathbf{a}^{(k)} = \lambda_k \mathbf{a}^{(k)}, \quad \mathbf{a}^{(k)} = (a_1^{(k)}, a_2^{(k)}, \ldots, a_N^{(k)}), \qquad (6)$$

$$(\hat{L}\mathbf{x})_i = \sum_j A_{ij}(x_i - x_j) = q_i x_i - \sum_j A_{ij} x_j. \qquad (7)$$

We assume, that a particle moves from vertex to vertex by hopping along edges. To every edge we ascribe a hopping rate w_{ij}, which is the probability to move from vertex j to vertex i per unit time. Hopping rates are assumed to be symmetric and equal to 1 for every edge, $w_{ij} = w_{ji} = A_{ij} = 0$ or 1. In this paper we fix w_{ij} but not the escape rate of a particle from a vertex, see Appendix A where other forms of a Laplace operator are listed. It turns out that our main conclusions are also valid if the escape rate from a vertex is fixed. This case will be discussed in detail in our next works. Assume that at $t=0$ the particle is at vertex j. Its motion is governed by the master equation for the propagator, which is the probability $p_{ij}(t)$ that at time t the particle is at vertex i,

$$\dot{p}_{ij}(t) = \sum_{k=1}^{N} [w_{ik} p_{kj}(t) - w_{ki} p_{ij}(t)] = \sum_{k=1}^{N} A_{ik} p_{kj}(t) - q_i p_{ij}(t). \qquad (8)$$

This equation is supplied with the initial condition $p_{ij}(0) = \delta_{ij}$. What is the value of the probability

$$\bar{p}_n(t) = \frac{1}{N} \left\langle \sum_{d(i,j)=n} p_{ij}(t) \right\rangle \qquad (9)$$

that at time t the particle is at distance $d(i,j) = n$ from a starting vertex? (The distance is the minimum shortest path between two vertices.) Here $\langle \cdots \rangle$ means the average over some statistical ensemble of graphs (over that of the configuration model in our case).

In the Laplace representation,

$$P_{ik}(s) = \int_0^\infty dt p_{ik}(t) e^{-st}, \qquad (10)$$

the propagator $\hat{P}(s)$ is the resolvent of the Laplace operator,

$$\hat{P}(s) = (s + \hat{L})^{-1}. \qquad (11)$$

[For the sake of clearness we note that $P_{ik}(s)$ is a matrix element of the operator $\hat{P}(s)$.] Consequently, the density of

eigenvalues is expressed in terms of the analytic continuations of the averaged values of the autocorrelator

$$\rho(\lambda) = \frac{1}{2\pi i} [\bar{P}_0(-\lambda - i0) - \bar{P}_0(-\lambda + i0)], \qquad (12)$$

where $\bar{P}_0(s) = \int_0^\infty dt \bar{p}_0(t) e^{-st}$. The inverse relation is

$$\bar{p}_0(t) = \int_{-i\infty+\delta}^{+i\infty+\delta} \frac{ds}{2\pi i} e^{st} \bar{P}_0(s) = \int_0^\infty d\lambda e^{-\lambda t} \rho(\lambda). \qquad (13)$$

III. MAIN EQUATIONS

We assume the thermodynamic limit, $N \to \infty$, and the fraction of vertices with a degree q, $N(q)/N \to \Pi(q)$. Here $\Pi(q)$ is a given degree distribution with a finite second moment, $\Sigma_q q^2 \Pi(q) < \infty$. In this limit, almost all finite subgraphs are trees, i.e., they have no closed loops within. The network is uncorrelated, i.e., degrees of any pair of vertices, connected or not, are independently distributed random variables. These features allowed us to describe the statistics of intervertex distances [29]. The problem under consideration is actually related to that work.

The equation for the resolvent of the Laplace operator (11) is

$$sP_{ik} = \delta_{ik} + \sum_j A_{ij}(P_{jk} - P_{ik}) = \delta_{ik} + \sum_j A_{ij} P_{jk} - q_i P_{ik}. \qquad (14)$$

Without lack of generality we choose the initial vertex $k=0$.

By definition, the nth connected component of a vertex i is a subgraph, containing all vertices j within the distance $d(i,j) \le n$ from the vertex i. For any finite n, in an infinite graph almost any nth connected component of vertex 0 is a tree. Actually, we analyze a random Bethe lattice. Degrees of its vertices are independent random variables. Its arbitrary chosen central vertex has the vertex distribution function $\Pi(q) = \langle \delta(q_0 - q) \rangle$. The other vertices have degree distributions equal among themselves but different from $\Pi(q)$. Noncentral vertex i of a degree q_i has one edge directed to the central vertex and $b_i = q_i - 1 \ge 0$ edges directed from it. Here b is the branching number of the vertex. Its distribution is given by

$$\Pi_1(b) = \frac{1}{2L} \sum_{i,j=1}^{N} \langle A_{ij} \delta(b_j - b) \rangle = \frac{(b+1)}{\bar{q}} \Pi(b+1), \qquad (15)$$

where $\bar{q} = 2L/N$,

$$L = \frac{1}{2} \sum_{i,j=1}^{N} A_{ij} = \frac{1}{2} \sum_{i=1}^{N} q_i \qquad (16)$$

is the total number of edges in the graph. In Eq. (15) we assume that $N \to \infty$, $L \to \infty$, but $2L/N \to \bar{q} = \Sigma_q q \Pi(q)$, where $\bar{q}$ is some finite number. $\Pi_1(b)$ is the probability that a randomly chosen end vertex of a randomly chosen edge in the graph has $b = q - 1$ edges apart from the chosen edge itself. It is convenient to use distributions Π and Π_1 in Z representation (see Appendix B).
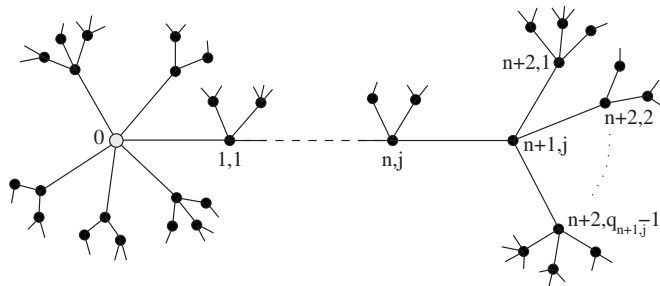
FIG. 3. Vertex $(n+1,i)$ at distance n from the starting vertex 0, its "ancestor" (n,j), and its "descendants" $(n+2,k)$. $k=1,2,\ldots,q_{n+1,k}-1$, $q_{n+1,k}$ is the degree of the vertex $(n+1,i)$.

Let (n,i) and $(n+1,j)$ be two vertices connected by an edge and at the distances n and $n+1$, respectively, from the starting vertex 0. We introduce the following random variable (see Fig. 3):

$$\tau_{n,ij}(s) = \frac{P_{n,i}(s) - P_{n+1,j}(s)}{P_{n,i}(s)}. \qquad (17)$$

It is obvious that the statistical properties of this variable are independent of the particular choice of vertex i in the nth shell of the initial vertex 0. The graph ensemble under consideration is completely defined by the degree distribution. All graphs with a given degree distribution have the same statistical weights. This, in particular, implies the statistical homogeneity of the ensemble. First, we randomly choose vertex 0. Second, we label all other vertices by two indices: The first one is the distance from vertex 0 (the shell's number), and the second index labels vertices within the shell. Third, we consider P_{ni}, which is the matrix element of the resolvent for the pair—vertex 0 and vertex i at distance n from vertex 0. It is a fluctuating random variable but its statistical properties are independent of the choice of i, because every averaging includes averaging over all vertices in shell n. The other fluctuating quantity in Eq. (17), $P_{n+1,j}$, is, of course, correlated with $P_{n,i}$. Nonetheless, due to the statistical independence of the vertices, this correlation is independent of the particular choice of the connected pair of vertices. Therefore, we can define the distribution function of $\tau_{n,ij}$, which is independent of i,j. In the Laplace representation this distribution is defined as

$$T_n(s,x) = \langle \exp[-x\tau_{n,ij}(s)] \rangle. \qquad (18)$$

Now let us recall that in the infinite network all finite connected components are trees. Moreover, in the thermodynamic limit the statistical properties of all τ variables are the same, i.e., they are independent of n too (see more detailed discussion in Appendix C). It implies the following important consequences. (i) In the thermodynamic limit, i.e., for an infinite network, $T_n \equiv T$ is independent of n. (ii) It is possible to obtain the closed equation for $T(s,x)$. (iii) The density of eigenvalues $\rho(\lambda)$, and, consequently, the autocorrelator $\bar{p}_0(t)$ can be expressed in terms of $T(s,x)$ (see Appendix C).

Equation for $T(s,x)$ may be written as

$$e^x T(s,x) = 1 + \sqrt{x} \int_0^\infty \frac{dy}{\sqrt{y}} I_1(2\sqrt{xy}) e^{-(1+s)x} \varphi_1[T(s,y)] \quad (19)$$

(see Appendix C), where I_1 is a modified Bessel function, and $\varphi_1(z)$ is the distribution of noncentral vertices branching numbers in Z-representation (see Appendix B). As Re $s>0$, this function has a solution with all properties of the Laplace transform of the distribution density of a non-negative random variable. This statement may be proved by using an approach of Ref. [30].

The function $\varphi_1(z)$ has the following properties: (i) $\varphi_1(0) = \Pi_1(1)$, which is the concentration of vertices with degree 1 ("dead ends"), and (ii) if the degree distribution $\Pi_1(q)$ decays slower than any exponent for $q \to \infty$, then $z=1$ is a point of singularity of $\varphi_1(z)$. The function $\varphi_1(z)$ in the complex plane is analytic within the circle $|z|<1$, $\varphi_1(1)=1$. The parameter $\varphi_1(0)$ is the crucial one in the division of the graph into connected components, see Appendix B. We show in Appendix B that the autocorrelator $\bar{P}_0(s)$ in the Laplace representation is given by

$$\bar{P}_0(s) = \int_0^\infty dx e^{-sx} \varphi[T(s,x)], \qquad (20)$$

where $\varphi(z)$ is the Z-transformation of the degree distribution $\Pi(q)$. The functions φ and φ_1 are connected as $\varphi_1(z) = \varphi'(z)/\varphi'(1)$, so that $\varphi(1)=\varphi_1(1)=1$. The density of eigenvalues $\rho(\lambda)$ and time-dependent autocorrelator $\bar{p}_0(t)$ can be obtained from Eqs. (12) and (13), respectively.

The propagator $\bar{p}_n(t)$ at $n>0$ may be expressed in terms of some functions $U_n(s,x)$, for which we have a linear recursion, relating U_n to U_{n-1} (see Appendix D). These functions are introduced in the following way. Let us choose two vertices (n,i) and $(n+1,j)$, connected by an edge (Fig. 3). We define $S^{(l)}_{n;i,j}(s)$ as

$$S^{(l)}_{n;i,j}(s) = \frac{1}{P_{n,i}(s)} \sum_{\{k\}} P_{n+l,k}(s). \qquad (21)$$

Here the summation is over all those vertices at a distance $n+l$ from vertex 0, whose shortest path to vertex 0 runs along the edge $(n,i) \to (n+1,j)$. In other words, the sum in Eq. (21) is over all vertices of the lth generation of the branch beginning from a chosen edge. For example, $S^{(1)}_{n;i,j}(s) = \sum_{k=1}^{b_{n+1,j}} P_{n+1,k}(s)$, as one can see from Fig. 3. Due to the statistical homogeneity of the network ensemble the statistical properties of random variables $S^{(l)}_{n;i,j}$ are independent of the choice of vertices i and j, if they are connected by an edge. For an infinite network, this statistic is also independent of n. The recursion relation can be derived for the following averaged quantity which depends only on l:

$$U_l(s,x) = \langle S^{(l)}_{n;i,j}(s) \exp[-x\tau_{n,ij}(s)] \rangle. \qquad (22)$$

The recursion relation is derived in Appendix D. It is of the following form:

$$U_l(s,x) = e^{-x} \int_0^\infty dy I_0(2\sqrt{xy}) e^{-(1+s)y} \varphi_1'[T(s,y)] U_{l-1}(s,y),$$

$$(23)$$

where I_0 is a modified Bessel function of zero order. This recursive relation is supplied with the initial condition

$$U_1(s,x) = \left(1 + \frac{\partial}{\partial x}\right) T(s,x).$$

$$(24)$$

Finally, the Laplace-transformed propagator $\bar{P}_l(s)$ is expressed as

$$\bar{P}_l(s) = \int_0^\infty dx e^{-sx} \varphi'[T(s,x)] U_l(s,x).$$

$$(25)$$

Equation (23) may be presented in the form

$$U_n = \hat{M} U_{n-1},$$

$$(26)$$

where $\hat{M}$ is a linear integral operator. Let $\mu_m^{-1}(s)$ and $\psi_m(s,x)$ be its eigenvalues and eigenfunctions, respectively,

$$\mu_m(s) e^{-x} \int_0^\infty dy I_0(2\sqrt{xy}) e^{-(1+s)y} \varphi_1'[T(s,y)] \psi_m(s,y) = \psi_m(s,x).$$

$$(27)$$

Note that the general theory of integral operators is usually formulated in terms of μ_m, which are called their characteristic numbers. Operator $\hat{M}$ becomes Hermitian after the substitution $\psi_m(x) = e^{sx/2} \{\varphi_1'[T(s,x)]\}^{-1/2} \chi(s,x)$. The kernel of the integral operator in Eq. (27) is bounded [31] if

$$\int_0^\infty dx \int_0^\infty dy I_0^2(2\sqrt{xy}) e^{-(2+s)(x+y)} \varphi_1'[T(s,x)] \varphi_1'[T(s,y)] < \infty.$$

$$(28)$$

This condition is always satisfied if $s>0$. Therefore, according to theorems about integral equations with a Hermitian bounded kernel [31], all eigenvalues of this operator are real and finitely degenerate. They form a discrete sequence bounded from below, without any condensation point except $\mu = \infty$. Eigenfunctions are orthogonal and normalizable with the weight function $e^{-sx} \varphi_1'[T(s,x)]$,

$$\int_0^\infty dx \varphi_1'[T(s,x)] e^{-sx} \psi_k(s,x) \psi_m(s,x) = \delta_{km}.$$

$$(29)$$

Hence, the solution of the recursive relation (23) may be presented as a series in the complete orthonormal set $\{\psi_m\}$,

$$U_l(s,x) = \sum_m A_m(s) \mu_m^{1-l}(s) \psi_m(s,x).$$

$$(30)$$

Taking into account the initial condition (24) and the orthonormality condition (29), the coefficients in this series may be written as

$$A_m(s) = \int_0^\infty dx \psi_m(x) \varphi_1'[T(s,x)] e^{-sx} \left(1 + \frac{\partial}{\partial x}\right) T(s,x).$$

$$(31)$$

Substituting Eq. (30) into Eq. (25), we obtain for $l>0$ the following relation:

$$\bar{P}_l(s) = \sum_m A_m(s) B_m(s) \mu_m^{1-l}(s),$$

$$(32)$$

where

$$B_m(s) = \int_0^\infty dx e^{-sx} \varphi'[T(s,x)] \psi_m(s,x).$$

$$(33)$$

The resulting propagator $\bar{P}_l(s)$ satisfies the condition of the conservation of the number of particles (signals), which in the Laplace representation is $\sum_{l=0}^\infty \bar{P}_l(s) = 1/s$. Taking into account Eq. (32) gives the following form of this condition:

$$\bar{P}_0(s) + \sum_m \frac{A_m(s) B_m(s)}{\mu_m(s) - 1} = \frac{1}{s}.$$

$$(34)$$

IV. CONTRIBUTION OF FINITE CONNECTED COMPONENTS

When the minimum vertex degree in the uncorrelated network $q_m \geq 2$, then (in the thermodynamic limit) almost all the network consists of one connected component. If, however, $q_m = 1$, i.e., $\Pi(1) = \Pi_1(1) = \varphi'(0) = \bar{q}\varphi_1(0) \neq 0$, then connected components constitute finite fraction of the network even in the thermodynamic limit. Their contribution to the propagator at $t \to \infty$ is obvious, and can be calculated in a straightforward way. We, however, find this contribution by using the technique described in Sec. III for the sake of illustration.

We set in $T(s,x)$ the limit $s \to 0$ and $x \to \infty$, with sx fixed, assuming that there exists a limiting function

$$\Theta(z) = \lim_{s \to \infty} T(s,z/s).$$

$$(35)$$

In Appendix E 3 we derive the following equation for Θ:

$$\Theta(z) = e^{-z} \varphi_1[\Theta(z)].$$

$$(36)$$

Comparing this equation with Eq. (B6) from Appendix B, one can conclude that $\Theta(z) = H(e^{-z})$, that is $\varphi[\Theta(z)] = \varphi[H(e^{-z})] = \langle \exp(-zM_i) \rangle$. Here M_i is the size of the connected component with a randomly chosen vertex i. It is obvious that the giant connected component, whose size is $\sim N$, does not contribute to $\Theta(z)$ at any $z>0$ as $N \to \infty$. From Eqs. (20) and (35), we obtain a clear result for the limiting value of the autocorrelator,

$$p_0^{(\text{eq})} \equiv \bar{p}_0(t = \infty) = \lim_{s \to 0} s\bar{P}_0(s) = \int_0^\infty dz \varphi[\Theta(z)] = \left\langle \frac{1}{M_i} \right\rangle.$$

$$(37)$$

It means that the equilibrium distribution of the signal is homogeneous within its connected component. Passing from

the variable z to $x=\Theta(z)$ and using Eq. (36), we calculate this integral,

$$p_0^{(eq)} = \varphi(t_c) - \frac{1}{2}t_c\varphi'(t_c). \quad (38)$$

This result also has a definite meaning,

$$p_0^{(eq)} = \frac{1}{N}\left\langle \sum_{i=1}^{N} \frac{1}{M_i} \right\rangle = \frac{1}{N}\left\langle \sum_{\text{Clusters}} 1 \right\rangle = \frac{N_c}{N}, \quad (39)$$

where N_c is the total number of finite connected components.

A nonzero equilibrium value of the autocorrelator indicates that the degeneracy of the Laplacian eigenvalue $\lambda=0$ is $\sim N$. The eigenvectors of this eigenvalue may be chosen in the following way. Each such eigenvector has unit vector components in one connected component and zeros in all others. The degeneracy is equal to the total number of connected components in the network.

The $t\to\infty$ contribution of finite connected components to $\bar{p}_l(t)$ at $l>0$ may be extracted from the functions

$$u_l(z) = \lim_{s\to\infty} U_l(s,z/s). \quad (40)$$

In this limit the recurrent relation (23) turns into

$$u_l(z) = e^{-z}\varphi_1'[\Theta(z)]u_{l-1}(z) \quad (41)$$

(see derivation in Appendix E 3). From Eq. (24) it also follows that $u_1(z)=\Theta(z)$, so

$$u_l(z) = \{e^{-z}\varphi_1'[\Theta(z)]\}^{l-1}\Theta(z). \quad (42)$$

Let us calculate $p_l^{(eq)}=\bar{p}_l(t=\infty)$. The stationary value of $P_{ij}(t)$ at $t\to\infty$ is equal to $1/M_j$, where M_j is a connected component of an initial vertex j. Consequently,

$$p_l^{(eq)} = \left\langle \frac{Q_j^{(l)}}{M_j} \right\rangle, \quad (43)$$

where $Q_j^{(l)}$ is the number of vertices at distance l from vertex j. Using Eqs. (25), (40), and (42), we obtain

$$p_l^{(eq)} = \int_0^\infty dx e^{-x}\Theta(x)\varphi'[\Theta(x)]\{e^{-x}\varphi_1'[\Theta(x)]\}^{l-1}. \quad (44)$$

At large l, the region $z\ll 1$, where $\Theta(z)$ is close to t_c, gives the main contribution to the integral in Eq. (44). As a result, at large l, we have

$$p_l^{(eq)} \approx \frac{b}{n}[\varphi_1'(t_c)]^n, \quad b = \frac{\bar{q}t_c^2[1-\varphi_1'(t_c)]}{\varphi_1'(t_c)[1-\varphi_1'(t_c)]+t_c\varphi_1''(t_c)}. \quad (45)$$

Here we used that $\Theta(0)=t_c=\varphi_1(t_c)$ and $-\Theta\prime(0)$ $=t_c/[1-\varphi_1\prime(t_c)]$, which follows from Eq. (36).

V. SPECTRAL DENSITIES AND PROPAGATORS FOR VARIOUS NETWORKS

Here we indicate four distinct kinds of uncorrelated random networks with qualitatively different asymptotic behav-

iors of $T(0,x)\equiv T_0(x)$ at $x\to\infty$, where $T_0(x)$ is the solution of Eq. (19) at $s=0$. At $x=0$, we always have $T_0(0)=1$. At $x\to\infty$ we have $T_0(\infty)=\lim_{x\to\infty}\lim_{s\to 0}T(s,x)=\Theta(0)$. These four types of networks differ from each other mainly by a value of the minimum vertex degree.

(1) If the minimum vertex degree $q_m\geq 3$, then identically $\Theta(x)=0$, and $T_0(x)$ exponentially decays to $T_0(\infty)=0$ (see Sec. V A).

(2) If $q_m=2$, then identically $\Theta(x)=0$, and $T_0(x)$ decays to $T_0(\infty)=0$, but slower than any exponent (see Sec. V B).

(3) If $q_m=1$, then there are two possibilities (see Sec. V C):

(a) If $z_1=\varphi_1'(1)>1$, then $0<\Theta(0)=t_c=\varphi_1(t_c)<1$, $T_0(x)\to t_c$ as $x\to+\infty$. In this case the graph has a giant connected component and a number of finite ones.

(b) If $z_1=\varphi_1'(1)<1$, then $\Theta(0)=1$. $T_0(x)=1$ as $x>0$. In this case the graph consists of only finite connected components.

Let us assume $q_m>1$ and consider in Eq. (19) the case of small positive s and large x. According to definition (18), $T(s,x)$ is actually a Laplace transform of the probability distribution of the non-negative random variable τ. Hence, it cannot decay at $x\to\infty$ faster than exponentially. In Appendix E we show that

$$T(s,x) \to A\exp[-\tau_m(s)x - \vartheta(s,x)]. \quad (46)$$

Here A is simply a constant, and ϑ is some correction term in the exponential. The coefficient τ_m at the main, linear in x, term in the exponential turns out to be the same as for regular Bethe lattice. It is defined by the relation

$$\frac{\tau_m}{1-\tau_m} = s + (q_m-1)\tau_m. \quad (47)$$

This equation has two real solutions as $s>s_c$, where

$$s_c = -\lambda_c = -q_m + 2\sqrt{q_m-1} \leq 0. \quad (48)$$

The physical branch of $\tau_m(s)$ is the one, positive at $s>0$. The other term in the exponential in Eq. (46) is a sublinear function of x. Namely,

$$\vartheta(x) = Bx^\alpha, \quad \alpha = \frac{\ln(q_m-1)}{2\ln[1/(1-\tau_m)]} = \frac{\ln[1/(1-\tau_c)]}{\ln[1/(1-\tau_m)]}, \quad (49)$$

where B is some constant. Here we introduced $\tau_c=\tau_m(s_c)$ $=1-1/\sqrt{q_m-1}$.

As s is close to s_c, α is close to 1, and $\vartheta(s,x)$ becomes comparable with the main term. It is this region that determines physically interesting results. The behavior of $\vartheta(s,x)$ at large x and s close to s_c determines the behavior of the spectral density $\rho(\lambda)$ near its edge $\lambda_c=-s_c$ and the behavior of the autocorrelator $\bar{p}_0(t)$ at large t. It turns out (see Appendix E), that the analytic continuation of $\vartheta(s,x)$ on negative $s<-\lambda_c$, $\vartheta(-\lambda,x)$, as a function of x is singular at some x_s $\sim 1/(\lambda-\lambda_c)$. Therefore, the upper limit of integration in Eq. (20) is in the upper half-plane of x, $\text{Im } x>0$ for $\text{Im } \lambda>0$ and vice versa. Then from Eq. (12) we obtain

$$\rho(\lambda) = \int_{-i\infty}^{+i\infty} \frac{dx}{2\pi i} e^{\lambda x} \varphi[T(-\lambda, x)]. \tag{50}$$

Note that if $q_m = 2$, then $s_c = \tau_c = 0$. We consider this case separately in Sec. V B.

A. Minimum degree $q_m > 2$

Let us set $T(s,x) = A \exp[-\tau_c x - \vartheta(s,x)]$. Note the difference of the definition of ϑ with that in Eq. (46): Here we have the term $-\tau_c x$ instead of $-\tau_m(s)x$ in the exponent. Therefore, $\vartheta'(s, x = +\infty) = \tau_m(s) - \tau_c \ll 1$ now is not equal to 0. In Appendix E 1 we obtain the following expression, valid when $x \gg 1$ and $|s - s_c| \ll 1$:

$$\vartheta(s,x) = \frac{x\sqrt{s - s_c}}{(q_m - 1)^{3/4}} \coth\left(\frac{\sqrt{s - s_c}\ln(Cx)}{(q_m - 1)^{1/4}\ln(q_m - 1)}\right), \tag{51}$$

where $C \sim 1$ is some number. Replace s by $-\lambda < -\lambda_c$, $\lambda_c = -s_c = 2\sqrt{q_m - 1} - q_m$. Then we have

$$\vartheta(-\lambda, x) = \frac{x\sqrt{\lambda - \lambda_c}}{(q_m - 1)^{3/4}} \cot\left(\frac{\sqrt{\lambda - \lambda_c}\ln(Cx)}{(q_m - 1)^{1/4}\ln(q_m - 1)}\right). \tag{52}$$

This function has a singularity when the argument of cot equal to π, i.e., at $x = x_0$, where

$$x_0 = C^{-1}\exp\left(\frac{\pi(q_m - 1)^{1/4}\ln(q_m - 1)}{\sqrt{\lambda - \lambda_c}}\right)$$

$$= C^{-1}(q_m - 1)^{\pi(q_m - 1)^{1/4}/\sqrt{\lambda - \lambda_c}}. \tag{53}$$

When x is close to x_0, one can replace $\cot z \to -1/(\pi - z)$ in Eq. (52).

Since $T(-\lambda, x)$ is small at large $0 < x \lesssim \pi$, one can replace $\varphi[T(-\lambda, x)]$ by its leading term $\Pi(q_m)\varphi^{q_m}T(-\lambda, x)$. Then, changing in Eq. (50) the integration variable, $x = x_0 y$, and taking into account Eq. (46), we obtain up to a factor ~ 1,

$$\rho(\lambda) \sim x_0 \int_C \frac{dy}{2\pi i} \exp\left[-x_0 y\left(b + \frac{a}{\ln y}\right)\right],$$

$$a = \frac{q_m \ln(q_m - 1)}{\sqrt{q_m - 1}}, \quad b = \frac{q_m - 2}{\sqrt{q_m - 1}}. \tag{54}$$

Finally, calculating this integral in the saddle point approximation, we obtain the density of eigenvalues of the Laplacian spectrum near its end point λ_c,

$$\rho(\lambda) \sim \exp\left[\frac{\beta}{2\sqrt{\lambda - \lambda_c}} - d\exp\left(\frac{\beta}{\sqrt{\lambda - \lambda_c}}\right)\right],$$

$$\beta = \pi(q_m - 1)^{1/4}\ln(q_m - 1), \tag{55}$$

where d is some constant. Substituting Eq. (55) into the expression for the autocorrelator (13) and using the saddle point approximation to calculate the integral, we obtain

$$\bar{p}_0(t) \sim \exp\left(-\lambda_c t - \frac{\beta^2 t}{\ln^2(dt)}\right). \tag{56}$$

Recall the notation $T_0(x) = T(0, x)$. Since $\tau_m(0) = (q_m - 2)/(q_m - 1) > 0$, we have $\varphi_1'[T_0(x)] \sim \exp[-(q_m - 2)\tau_m(0)x]$ at $x \to +\infty$, and the kernel of the integral equation (27) satisfies the condition (28). It implies that at $s = 0$ in the discrete sequence of characteristic numbers $\mu_m(0) \equiv \mu_m$, there is the minimum one, $\mu_0 > 0$. In Appendix F we show that (i) $\mu_0 = 1$, (ii) this characteristic number is the minimum one, and (iii) the corresponding normalized eigenfunction is

$$\psi_0(0, x) \equiv \psi_0(x) = -d_0 T_0'(x), \quad d_0$$

$$= \left(\int_0^\infty dx \varphi_1'[T_0(x)]T_0'^2(x)\right). \tag{57}$$

Here d_0 ensures proper normalization (29), and the minus sign stands simply for convenience ensuring $\psi_0(x) \geq 0$.

When $s > s_c$, in particular, near $s = 0 > s_c$, the kernel in the integral equation (27) is well behaved, and all $\mu_m(s)$ are analytic functions of s. We can leave in Eq. (32) only the leading term with the minimum μ_m. Then, $\bar{P}_l(s) \approx A_0(s)B_0(s)\mu_0^{1-l}(s)$ for large distances l from the initial vertex. So at large time t and large distance l, the propagator $\bar{p}_l(t)$ is approximately

$$\bar{p}_l(t) = \int_{-i\infty}^{+i\infty} \frac{ds}{2\pi i} e^{st}\bar{P}_l(s) \approx \int_{-i\infty}^{+i\infty} \frac{ds}{2\pi i} e^{st}A_0(s)B_0(s)\mu_0^{1-l}(s). \tag{58}$$

If the expression under the integral is analytic in s along the integration contour, the main contribution to the asymptotic of the integral gives the vicinity of the saddle point, where $st - l\ln\mu_0(s)$ is maximal. The saddle point position s_c is the solution of the equation $t = l\mu_0'(s)/\mu_0(s)$. As a result, we have

$$\bar{p}_l(t) \approx \frac{1}{\sqrt{2\pi\beta(s_c)l}}A_0(s_c)B_0(s_c)\mu_0(s_c)\exp[s_c t - l\ln\mu_0(s_c)], \tag{59}$$

where $\beta(s_c) = [\ln\mu_0(s)]''|_{s=s_c}$. At a given $t \gg 1$, this expression has a maximum as a function of l at $l = l_m(t)$, where

$$\frac{\partial}{\partial l}[s_c t - l\ln\mu_0(s_c)] = \ln\mu_0(s_c) = 0,$$

i.e., where $\mu_0[s_c(l_m, t)] = 1$. Here $s_c(l, t)$ is defined from the saddle point condition. Since $\mu_0(0) = 1$, the propagator $\bar{p}_l(t)$ is maximal at $l = l_m$, $s_c(l_m, t) = 0$. The behavior of $\mu_0(s)$ at small values of $|s|$ determines the shape of the propagator near its maximum point. Since $\mu_0(s)$ is an analytic function near $s = 0$, and $\mu_0(0) = 1$, one can write $\ln\mu_0(s) = \alpha s - \beta s^2/2 + \cdots$ and replace $A_0(s)$ and $B_0(s)$ by $A_0(0)$ and $B_0(0)$. Then, the expression (59) is reduced to a Gaussian integral, and we have

$$\bar{p}_l(t) \approx \frac{A_0(0)B_0(0)}{\sqrt{2\pi\beta l}}\exp\left(-\frac{(t-\alpha l)^2}{2\beta l}\right). \qquad (60)$$

On the left-hand side of the normalization condition (34), only the term with $m=0$ has a simple pole singularity at $s=0$. Then, we have $\lim_{s\to 0} A_0(s)B_0(s)/[\mu_0(s)-1] = A_0(0)B_0(0)/\alpha = 1$. We substitute the expressions for $A_0(0)$ and $B_0(0)$ from Eqs. (31) and (33), where the function $\psi_0(x)$ is expressed in terms of $T_0(x)$ by using Eq. (57). This leads to

$$\alpha \equiv v^{-1} = A_0(0)B_0(0)$$

$$= \frac{\int_0^\infty dx T_0'(x)\varphi_1'[T_0(x)]\left(1+\dfrac{d}{dx}\right)T_0(x)\int_0^\infty dx\varphi'[T_0(x)]T_0'(x)}{\int_0^\infty dx\varphi_1'[T_0(x)]T_0'^2(x)}.$$

$$(61)$$

The parameter $\beta \sim 1$ must be positive to ensure the convergence in the summation over l. Equation (60), as one can see from its derivation, is valid if the saddle point position $|s_c| = |(t-l/v)/(\beta l)| \ll 1$. So we may replace βl in Eq. (60) with its value at $l=l_m$, $\beta v t$, and, finally,

$$\bar{p}_l(t) \approx \frac{1}{\sqrt{2\pi Dt}}\exp\left(-\frac{(l-vt)^2}{2Dt}\right), \qquad (62)$$

where $D=\beta v^3$. Despite that our network is random, a signal spreads over the network as a Gaussian packet, moving with the constant velocity v from an initial vertex, and with the dispersion $\overline{(l-l_m)^2}$, which grows linearly with time. This is the same kind of evolution as on a regular Bethe lattice.

Equation (62) is valid when one can neglect terms of the order of s^3 and higher in the expansion of $\ln\mu_0(s)$ in the powers of s, i.e., when $l|s|_c^3 \sim t|s|_c^3 \ll 1$. Since $s_c=(l/v-t)/l \sim (l-vt)/t$, this condition is reduced to $|l-vt| \ll t^{2/3}$. The width of the packet is $\sim t^{1/2} \ll t^{2/3}$, and so expression (62) is relevant.

B. Minimum degree $q_m=2$

If $q_m=2$, then $s_c=0$ as one can see from Eq. (48). That is, $T(s,x)$ as a function of x becomes nonanalytic at $s<0$. Besides, $\tau_c=\tau_m(s_c)=0$, so that the decay of $T_0(x)\equiv T(0,x)$ is nonexponential in contrast to $q_m>2$. Setting $T(s,x)=\exp[-\vartheta(s,x)]$, we obtain the following expression for small s and large $x>0$ (see Appendix E 2):

$$\vartheta(s,x) \approx \frac{1}{\sqrt{s}}\left(\sqrt{sx(a/\pi+sx)}+\frac{a}{\pi}\text{arcsinh}\sqrt{\frac{\pi sx}{a}}\right)$$

$$+\frac{1}{4}\ln\left(s+\frac{a}{\pi x}\right)+C, \quad a=\pi\ln\left(\frac{\bar{q}}{2\Pi(2)}\right)>0,$$

$$(63)$$

where $C \sim 1$ is some constant. In the following we omit numerical constants as inessential. When analytically continued to $s=-\lambda<0$, $\vartheta(-\lambda,x)$ as a function of x acquires a singularity at $x=x_c=a/\pi\lambda$. The density of Laplacian eigenvalues,

$\rho(\lambda)$, can be obtained from Eq. (50). The main contribution to the integral in Eq. (50) arises from the close vicinity of the singularity point. In Eq. (50), we expand ϑ near x_c in the integral and change the integration variable from x to $\zeta = \lambda(x_c-x)$. This results in

$$\rho(\lambda) \sim \frac{1}{\lambda^{3/2}}\exp\left(-\frac{a}{\sqrt{\lambda}}\right)\int_{C'}\frac{d\zeta}{2\pi i}\zeta^{-1/2}\exp\left(-\zeta+\frac{4\zeta^{3/2}}{3\sqrt{\lambda}}\right).$$

Here the integral is $\sim\lambda^{1/6}$, and the asymptotics at $0<\lambda\ll 1$ is

$$\rho(\lambda) \sim \frac{1}{\lambda^{4/3}}\exp\left(-\frac{a}{\sqrt{\lambda}}\right). \qquad (64)$$

We substitute this expression into Eq. (13), and by using the saddle point approximation, arrive at the following long t asymptotics for the autocorrelator:

$$\bar{p}_0(t) \sim t^{1/18}\exp\left[-3\left(\frac{a}{2}\right)^{2/3}t^{1/3}\right]. \qquad (65)$$

Let us now consider the propagator $\bar{p}_l(t)$ at $l\gg 1$, $t\gg 1$. This asymptotics is also defined by Eq. (58). As for $q_m>2$, the main contribution to the integral is from the region of small $|s|$. The difference is that here $A_0(s)$, $B_0(s)$, and $\mu_0(s)$ all have a singularity at $s=0$. Namely, $s=0$ is a branching point, giving a cut along the line $(0,-\infty)$ in the complex plane of the variable s. We will show, however, that this singularity is very weak and does not contribute essentially to the propagator, except for relatively small distances l.

Indeed, the small s, large x asymptotics of the eigenfunction $\psi_0(s,x)$, corresponding to the largest characteristic number $\mu_0(s)=1+o(s)$, is (see Appendix E 2)

$$\psi_0(s,x) \approx x^{-1/2}T(s,x)$$

$$\sim x^{-1/2}\left(s+\frac{a}{\pi x}\right)^{-1/4}\exp\left[-\frac{1}{\sqrt{s}}\left(\sqrt{sx(a/\pi+sx)}\right.\right.$$

$$\left.\left.+\frac{a}{\pi}\text{arcsinh}\sqrt{\pi sx/a}\right)\right]. \qquad (66)$$

Then, comparing the leading terms in Eqs. (31) and (33) with that in Eq. (20), we conclude that the asymptotics of $\text{Im }A_0(-\lambda)$ and of $\text{Im }B_0(-\lambda)$ on λ are nearly the same as that of $\rho(\lambda) \sim \text{Im }\bar{P}_0(-\lambda)$. The difference is in powers of λ in the preexponential factors. In the leading order,

$$\text{Im}[A_0(-\lambda)] \sim \text{Im}[B_0(-\lambda)]$$

$$\sim \int_{-i\infty}^{+i\infty}\frac{dx}{2\pi i}e^{\lambda x}T(-\lambda,x)\psi_0(-\lambda,x)$$

$$\sim \lambda^{-5/6}\exp\left(-\frac{a}{\sqrt{\lambda}}\right). \qquad (67)$$

The rate of singularity of $\mu_0(s)$, if measured as a jump of a function across the cut near its branching point, is even smaller than in Eq. (67) for small $\lambda=-s>0$. Let us take the eigenfunction equation (27) at $m=0$, setting $x=0$. Then we have

$$\mu_0^{-1}(s)\psi_0(s,0) = \int_0^\infty dy e^{-(1+s)y}\varphi_1'[T(s,y)]\psi_0(s,y).$$

Then, setting $s = -\lambda < 0$, and properly deforming the integration contour, we obtain in the leading order

$$\text{Im } \psi(-\lambda,0) - \psi(0,0)\text{Im } \mu_0(-\lambda)$$
$$\sim \int_C \frac{dx}{\sqrt{x}}\exp\left(-(1-\lambda)x - \frac{1}{\sqrt{\lambda}}\vartheta(-\lambda,x)\right),$$

where the function ϑ is given by Eq. (63). As a function of x this integral has a singularity at $x = a/\pi\lambda$. In comparison with the integral for $\rho(\lambda)$, the above integral has an additional term $-x$ in the exponent, which turns into $-a/\pi\lambda$ at the singularity point. Therefore, we estimate the singularity of μ_0 near $s = 0$ as $\text{Im } \mu_0(-\lambda) \sim \exp(-a/\pi\lambda)$.

Since all multipliers in Eq. (58) have sufficiently weak singularities, we replace $A_0(s)$ and $B_0(s)$ with their values at $s = 0$ and neglect the singular part of $\mu_0(s)$, leaving only the regular part of the expansion $\ln \mu_0(s) = s/v + \beta s^2/2 + \cdots$. As a result, we arrive at the same Gaussian expression for the propagator, Eq. (62).

If we, however, fix the distance $l \gg 1$ and increase the time t, the saddle point $s_c < 0$ in the integral (58) moves farther in the direction of negative s, and at large enough t the contribution of the singularity becomes essential. Deforming contour of integration, we rewrite Eq. (58) in the following form:

$$\bar{p}_l(t) \approx \frac{1}{\pi}\int_0^\infty d\lambda e^{-\lambda t}\text{Im}[A_0(-\lambda)B_0(-\lambda)\mu_0^{1-l}(-\lambda)]. \tag{68}$$

$\text{Im } \mu_0 \sim \exp(-a/\pi\lambda)$ is small compared to $\text{Im } A_0(-\lambda) \sim \text{Im } B_0(-\lambda) \sim \exp(-a/\sqrt{\lambda})$. So we neglect the singularity of μ_0 and set $\ln \mu_0(-\lambda) = -\lambda/v$. Thus we arrive at

$$\bar{p}_l(t) \sim \int_0^\infty \frac{d\lambda}{\lambda^{5/6}}\exp\left[-\lambda\left(t - \frac{l}{v}\right) - \frac{a}{\sqrt{\lambda}}\right]. \tag{69}$$

Calculating the integral in the saddle point approximation we obtain

$$\bar{p}_l(t) \sim \left(t - \frac{l}{v}\right)^{-5/18}\exp\left[-3\left(\frac{a}{2}\right)^{2/3}\left(t - \frac{l}{v}\right)^{1/3}\right]. \tag{70}$$

Expanding $\ln \mu_0(s)$, we neglected terms of the order of s^2 and higher. This is justified if the saddle point position in the integral (69), $\lambda_s \sim (t - l/v)^{-2/3}$, obeys the condition $l\lambda_s^2 \ll \lambda_s^{-1/2}$ which is equivalent to $t - l/v \gg t^{3/5}$. Otherwise, $\bar{p}_l(t)$ is given by Eq. (62), which means that the probability for the signal to return is small. This form of the packet tail is due to the possibility that either initial vertex 0 or the final one in the lth shell may occur in a chain fragment in the graph.
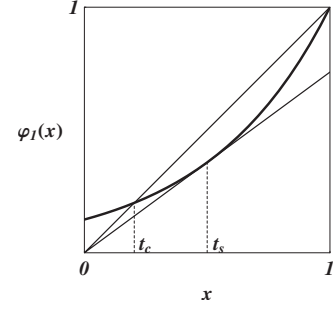

FIG. 4. Graphical solution of the equations for t_c, $t_c = \varphi_1(t_c)$ and for $t_s > t_c$, $\varphi_1'(t_s) = \varphi_1(t_s)/t_s$. The thick solid line shows the function $\varphi_1(x)$. The thin solid lines represent the function x and the tangent to the curve $\varphi_1(x)$, which passes through the center of coordinates.

C. Minimum degree $q_m = 1$

When there is a finite fraction of "dead ends," i.e., vertices of degree 1, the network contains finite-size connected components. They lead to the δ-functional peak in the Laplace spectrum and so to nonzero limits of the averaged propagators at $t \to \infty$. If $\varphi_1'(1) > 1$ (Appendix B), then besides the finite connected components, there is a giant connected one whose size scales as the network size. Here we show that the contribution of this giant connected component to the observable quantities is qualitatively the same as in networks with $q_m = 2$.

If $q_m = 1$, Eq. (19) still has the nontrivial solution $T_0(x) \equiv T(0,x)$. $T_0(0) = 1$ as for any other q_m, but $T_0(+\infty) = \lim_{x\to 0}\lim_{s\to 0} T(s,x/s) = \Theta(0) = t_c > 0$ (see Appendix B). At small $s > 0$ and large $x > 0$, the function $T(s,x)$ is close to $\Theta(sx)$, and so we search for $T(s,x)$ in the following form:

$$T(s,x) = \Theta(sx) + e^{-\vartheta(s,x)}, \tag{71}$$

where the last term is assumed to be small. The asymptotic solution for ϑ is (Appendix E 3)

$$\vartheta(s,x) = \frac{1}{\sqrt{s}}g(sx) + \frac{1}{4}\ln sg'^2(sx) + C,$$

$$g(z) = \int_0^z dy \sqrt{1 - \frac{\ln \varphi_1'[\Theta(y)]}{y}}, \tag{72}$$

where $C \sim 1$ is some constant. Continuing this result to $s = -\lambda > 0$, we take into account that $g(z)$ has a singularity at $z = z_s < 0$, where z_s satisfies the equation $1 - \ln \varphi_1'[\Theta(z_s)]/z_s = 0$. The equation for z_s, $\varphi_1'[\Theta(z_s)] = e^{z_s}$, becomes more comprehensive with a new variable $t_s = \Theta(e^{-z_s})$. Using the implicit definition (36) of $\Theta(z)$, we arrive at the equation for t_s,

$$\varphi_1'(t_s) = \frac{\varphi_1(t_s)}{t_s}. \tag{73}$$

This equation is shown graphically in Fig. 4, together with the equation for $t_c = \varphi_1(t_c) < t_s$, $\Theta(t_c) = 0$.

An expression for the spectral density $\rho(\lambda)$ may be obtained by calculating the integral in Eq. (50), where for small λ and large x, we approximately set

$$\varphi[T(-\lambda,x)] \approx \varphi[\Theta(-\lambda x)] + \varphi'[\Theta(-\lambda x)]e^{-\vartheta(-\lambda,x)}. \tag{74}$$

While the first term results in the δ-functional peak, the integration of the last one gives the asymptotics of $\rho(\lambda)$ at small positive λ. The main contribution into the integral gives the close vicinity of the positive singularity point $x_s = -z_s/\lambda > 0$. Near this point the first multiplier in the last term in Eq. (74) can be replaced by a constant $\varphi'[\Theta(z_s)] = \varphi'(t_s)$, and the function g in the expression for ϑ in (72) can be replaced by its expansion $g(z_s + \eta) = ia + ih\eta^{5/4}$, $h \sim 1$, $|\eta| \ll 1$ with

$$a = \int_{t_c}^{t_s} dx \frac{\varphi_1(x) - x\varphi_1'(x)}{x\varphi_1(x)} \sqrt{\frac{\ln \varphi_1'(x)}{\ln[\varphi_1(x)/x]} - 1} \tag{75}$$

(see Appendix E 3). Then we arrive at the following asymptotic result for $\rho(\lambda)$:

$$\rho(\lambda) - p_0^{(eq)}\delta(\lambda) \sim \lambda^{-9/10}\exp\left(-\frac{a}{\sqrt{\lambda}}\right). \tag{76}$$

As it follows from Eqs. (76) and (13), the autocorrelator $\bar{p}_0(t)$ decays to its equilibrium value as

$$\bar{p}_0(t) - p_0^{(eq)} \sim \int_0^\infty \frac{d\lambda}{\lambda^{9/10}}\exp\left(-\lambda t - \frac{a}{\sqrt{\lambda}}\right)$$
$$\sim t^{-7/30}\exp\left[-3\left(\frac{a}{2}\right)^{2/3}t^{1/3}\right]. \tag{77}$$

One can calculate the propagator $\bar{p}_l(t)$ at large l using Eq. (58). As compared with Secs. V A and V B, the kernel of Eq. (27) is not any more bounded at $s=0$ because the integral in Eq. (28) becomes divergent. Due to this fact, the spectrum of Eq. (27) contains a continuous part. Let us find eigenvalues μ_ξ and eigenfunctions $\psi_\xi(x)$ in the continuous spectrum. [The notation $\psi_m(x) \equiv \psi_m(s=0,x)$ we leave for the discrete part of the spectrum.] We see in Sec. IV that when $s \to 0$, the recursion relation (23) can be transformed to Eq. (41), assuming that $U_l(s,x) \approx u_l(sx)$ at small s. An equation for the eigenfunctions is

$$\psi_\xi(x) = \mu_\xi e^{-x}\varphi_1'[\Theta(x)]\psi_\xi(x),$$

which has the solutions $\psi_\xi(x) = \delta(x-\xi)$ corresponding to the eigenvalues $\mu_\xi = e^\xi/\varphi_1'[\Theta(x)]$. It is the continuous part of the spectrum that after proper modification of the relations (29)–(33), gives the stationary part of propagator (44). Suppose that there is a giant connected component in the network. Then along with the continuous part of the spectrum, whose minimum characteristic number is $\mu_{\xi=0} = 1/\varphi_1'(t_c) > 1 = \mu_{m=0}$, there is a discrete spectrum with the minimum characteristic number $\mu_0(s=0) = 1$ corresponding to the eigenfunction $\psi_0(x) = -T_0'(x)$ (see Appendix F).

In the same way as for $q_m = 2$ (see Appendix E 3), one can show that the asymptotics at small s and large positive x of the eigenfunction $\psi_0(s,x)$, corresponding to the lowest characteristic number $\mu_0(s)$, is $\psi_0(s,x) \sim x^{-1/2}\exp[-\vartheta(s,x)]$, where $\vartheta(s,x)$ is given by Eq. (72).

From Eqs. (31) and (33) we obtain

$$\text{Im}[A_0(-\lambda)] \sim \text{Im}[B_0(-\lambda)] \sim \int_{-i\infty}^{+i\infty}\frac{dx}{2\pi i}e^{\lambda x}\psi_0(-\lambda,x)$$
$$\sim \int_{-i\infty}^{+i\infty}\frac{dx}{\sqrt{x}}e^{\lambda x - \vartheta(-\lambda,x)} \sim \lambda^{-1/2}\exp\left(-\frac{a}{\sqrt{\lambda}}\right). \tag{78}$$

This equation differs from Eq. (67), because in the singularity point $x_s = -z_s/\lambda \gg 1$, $T(-\lambda,x_s) \approx \Theta(-z_s) \sim 1$. So we omitted $T(-\lambda,x_s)$ in Eq. (78), in contrast to the case $q_m = 2$, where the function $T(-\lambda,x_s)$ is of the same order of smallness as $\psi_0(-\lambda,x_s)$. Here, as for $q_m = 2$, the singularity of $\mu_0(s)$ is such that the jump along the cut $(-\infty,0)$ in the complex planes s behaves as $\text{Im}\,\mu_0(-\lambda) \sim \exp(-a/\lambda)$.

The derivation of $\bar{p}_l(t)$ for $q_m = 1$ is similar to that for $q_m = 2$. We arrive at the same moving Gaussian packet (62). The only difference is that now we must take into account the contribution of finite clusters (continuous spectrum). The results for $l \gg 1$ and $t \gg 1$ are

$$\bar{p}_l(t) = p_l^{(eq)} + \frac{1}{\sqrt{2\pi Dt}}\exp\left(-\frac{(l-vt)^2}{2Dt}\right) \tag{79}$$

for $|vt - l| \ll t^{3/5}$, where v is given by Eq. (45), and $D = v^3\mu_0''(0)$. In the low l tail, $l < vt$, $vt - l \gg t^{3/5}$, the form of the propagator is modified to

$$\bar{p}_l(t) - p_l^{(eq)} \sim \left(t - \frac{l}{v}\right)^{-13/30}\exp\left[-3\left(\frac{a}{2}\right)^{2/3}\left(t - \frac{l}{v}\right)^{1/3}\right]. \tag{80}$$

Thus, again, we have the Gaussian packet, Eq. (79), moving within the giant connected component. This Gaussian is supplied with a small tail at $1 \ll l \ll t$, Eq. (80). The reason for this tail is that initial or final vertices may be "dead ends."

VI. SUMMARY, DISCUSSION, AND CONCLUSIONS

In this paper we have presented a theory, which enables us with the analytical calculation of statistical properties of the Laplacian operators of infinite random networks and random walks on them. We have considered the resolvent of the Laplacian and the propagator of a random walk. These characteristics are connected through a Laplace transform, Eqs. (10) and (11). In particular, the average values of the diagonal element of the resolvent matrix give us the spectral density of the Laplacian, Eq. (12), and the time dependence of the autocorrelator. We have also derived equations, which solution allows us to find the averages of the nondiagonal elements of the resolvent. After the Laplace transformation, these averages show how the distance of the signal from its origin changes with time, Eq. (9).

Our scheme is based on equations relating the distributions (or other statistical properties) of random variables. This is an essential advantage over most of the existing approaches, based on equations, which relate to the values of some random variables for a given network realization. To

solve the problems of the Laplacian spectrum and of random hopping motion, one must take the following steps.

(i) Solve the integral equation (19) for the function $T(s,x)$ defined by Eqs. (17) and (18). [In the equivalent form, it is Eq. (C7).] Technically, it is the most difficult step. We have only obtained the asymptotics of $T(s,x)$ at Re $x \to +\infty$. We have found that T as a function of x is an analytic and exponentially decaying function as $s > s_c$, where $s_c = -\lambda_c \leq 0$ is a parameter which depends only on the minimum vertex degree q_m.

(ii) With $T(s,x)$, one can (a) calculate the average of the resolvent's diagonal, Eq. (20), then (b) analytically continue the result from the positive s to $s = -\lambda \pm i0$, $\lambda > 0$, and finally (c) obtain, using Eq. (12), the spectral density of the Laplacian [32].

(iii) With the known $\rho(\lambda)$ near the spectrum edge, obtain the asymptotics of the autocorrelator $\bar{p}_0(t)$ at $t \to \infty$ by using Eq. (13).

(iv) Find the sequence of functions $U_l(s,x)$, $l \geq 1$, definition (22), by using the integral recursive relation (23) with the initial condition (24). Then obtain the Laplace-transformed propagator $\bar{P}_l(s)$ by calculating the integral (25) [33].

(v) Calculate the inverse Laplace transform of $\bar{P}_l(s)$, that is, the propagator $\bar{p}_l(t)$. The asymptotics of $\bar{p}_l(t)$ at large l and large t is determined by the smallest characteristic number $\mu_0(s)$ at small $|s|$.

The results of these calculations of asymptotics are summarized in Table I and Fig. 1. If $q_m \geq 3$, the tail in the density of eigenvalues decreases extremely rapidly with $1/(\lambda - \lambda_c)$, see Eq. (55), and therefore practically cannot be revealed by numerical methods. Studies based on these methods usually result in a form of $\rho(\lambda)$ resembling Wigner's semicircle law (see, e.g., Refs. [4,6]). This is also the case in networks with $q_m = 1, 2$. In particular, for the Erdős-Rényi graphs our results contain the leading contributions to the spectra and the asymptotics of the random walk autocorrelator found in Ref. [3]. In addition to these leading terms, we have found power-law factors, see Eqs. (65) and (77). In Ref. [8], Laplacian spectra of an infinite uncorrelated network with high density of connections (the mean degree is assumed to be $\bar{q} \to \infty$) were found. The authors of that work used the static model which provides a network with, in particular, vertices of degree 1 and 2. Nonetheless, the spectrum of the Laplacian operator in Ref. [8] had a gap. The reason for this difference from our results is not the different model of an uncorrelated network but rather the high mean degree limit assumed in the work [8].

When are our analytical results observable? Let us inspect the resulting expressions for the propagator $\bar{p}_l(t)$. Our results are based on the tree ansatz, $p_l(t)$ should have nonzero values in the small (compared to the whole network) vicinity of the starting vertex 0, so that we can treat this region as a tree. At large times t the signal spreads at the distance $\bar{l} = vt \sim t$, Eq. (62). The mean intervertex distance in the network is $\sim \ln N$ [29,35]. So, our results are applicable if $1 \ll t \ll \ln N$. In networks with $q_m \geq 3$ the decay of the autocorrelator is basically exponential with some correction [see Eq. (56)]. This correction can be observed if

$$1/\ln^2 t \sim 1/\ln^2 \ln N \ll 1. \tag{81}$$

It is impossible to fulfill this criterion either in real-world networks or in numerical simulations.

In the networks, containing chainlike segments, i.e., when $q_m = 1$ or 2, the criterion is much less stringent. We require that the value of the autocorrelator $\bar{p}_0(t)$ in (65) at the characteristic time $t \sim \ln N$, essentially exceeds its equilibrium value $\bar{p}_0(t=\infty) \sim 1/N$ for a finite network. So in these networks, our dependences are observable if

$$t^{1/3}/\ln N \sim \ln^{-2/3} N \ll 1, \tag{82}$$

which is much easier to satisfy than condition (81).

In many applications of the Laplacian spectrum, results, obtained in the infinite network limit, are of little use. A good example is synchronization [9,34]. In this problem the lowest, size-dependent eigenvalue of the Laplacian plays a key role. Let us briefly discuss the role of this eigenvalue in application to our problems. The process of a signal spread over the network consists of two distinct stages. We discussed the first one. In the second stage, the essence of the process is the relaxation to the homogeneous distribution, where the probability to find a signal at any vertex is the same, namely, $1/N$. In this last stage, $|p_{i0}(t) - 1/N| \ll 1/N$. In this situation, loops must be taken into account. Furthermore, in this stage, the knowledge of the Laplacian spectral density is not sufficient. Rather, one should ask the following: What is the probability distribution of λ_2 (the lowest nonzero eigenvalue)?

Figure 1 demonstrates that when $q_m = 1$ or 2, there is no gap in the Laplace spectrum. On the other hand, we have shown that in infinite networks with minimum vertex degree $q_m > 2$, the density $\rho(\lambda) = 0$ for $0 < \lambda < \lambda_c$. In contrast, in finite networks, Laplacian eigenvalues λ_i exist in this range, though only a very small fraction of the total number of the eigenvalues. The statistics of this part of the spectrum determines the second stage of the evolution of $p_{i0}(t)$ to the equilibrium. We believe that this stage may be described in the framework of an approach developed in Ref. [29] for calculation of intervertex distance distributions. We leave this challenging problem for future study.

In summary, we have strictly shown that the region of low eigenvalues in the Laplacian spectra of sparse uncorrelated complex networks and the asymptotics of random walks on them are essentially determined by the lowest vertex degree in a network.

ACKNOWLEDGMENTS

APPENDIX A: OTHER LAPLACIANS AND PROCESSES

Here we give a short list of standard Laplacians and corresponding processes. Three different forms of a Laplacian

operator are discussed in literature. In this paper we discussed the form (1) corresponding to the process defined by Eq. (2). The second form

$$L_{ij} = \delta_{ij} - \frac{1}{q_i} A_{ij}, \tag{A1}$$

corresponds to the following process:

$$\dot{p}_{ij}(t) = \sum_{k=1}^{N} \frac{1}{q_k} A_{ik} p_{kj}(t) - p_{ij}(t), \quad p_{ij}(0) = \delta_{ij}. \tag{A2}$$

This is a random walk process with the unit escape rate of a particle from any vertex. The particle jumps to any of q_i nearest neighbors of vertex i with the same probability $1/q_i$. We do not consider this process here, although it can be described in the framework of the approach of this paper. We have found that the singularity of the spectrum at the lowest eigenvalue of this Laplacian and the long-time asymptotics of the autocorrelator of this random walk are quite similar to those we found for the operator (1) and the process (2).

The third, "normalized," form

$$L_{ij} = \delta_{ij} - \frac{1}{\sqrt{q_i q_j}} A_{ij} \tag{A3}$$

(see, e.g., Ref. [6]) is, one may say, equivalent to the form (A2) in the following sense. Operators (A2) and (A3) are connected by a similarity transformation. The connecting operator $\hat{W}$ is diagonal, $W_{ij} = \delta_{ij} q_i$. These two operators have the same spectrum of eigenvalues. Their eigenfunctions are connected by the operator $\hat{W}$.

APPENDIX B: DEGREE DISTRIBUTION IN Z-REPRESENTATION

Here we represent the main relations in the Z-transformation (generating function) approach to uncorrelated networks. This useful technique allows one to completely describe the structure of these random networks. The Z-representation of a discrete random variable $q_i = 0, 1, 2, \dots$ is defined as

$$\varphi(z) = \frac{1}{N} \sum_{i=1}^{N} \langle z^{q_i} \rangle = \sum_{q=0}^{\infty} \Pi(q) z^q. \tag{B1}$$

$\varphi(z)$ is also called the generating function of $\Pi(q)$. It is obvious that $\varphi(1) = 1$. Differentiating $\varphi(z)$ and setting $z=1$, we obtain an expression for the average vertex degree,

$$\varphi'(1) = \sum_{q=0}^{\infty} q \Pi(q) = \langle q \rangle \equiv \bar{q}. \tag{B2}$$

In general,

$$\left(x \frac{d}{dz} \right)^m \varphi(z) \bigg|_{x=1} = \sum_{q=0}^{\infty} q^m \Pi(q) = \langle q^m \rangle. \tag{B3}$$

For branching numbers $b_i = q_i - 1$, we have

$$\varphi_1(z) = \frac{1}{2L} \sum_{i,j=1}^{N} \langle A_{ij} z^{b_j} \rangle = \frac{1}{2L} \sum_{j=1}^{N} \langle q_j z^{q_j - 1} \rangle$$

$$= \frac{1}{\bar{q}} \sum_{q=0}^{\infty} q \Pi(q) z^{q-1} = \frac{\varphi'(z)}{\bar{q}}. \tag{B4}$$

The function φ_1 also obeys a normalization condition, $\varphi_1(1) = 1$.

This function was successfully used by Newman, Strogatz, and Watts [35] (compare with the earlier works by Molloy and Reed [36,37]) in their calculations of the size distributions of nth connected components of a vertex. Recall that this is a number of vertices which are not further than n steps from a vertex. For example, the distribution for the first connected component in Z representation is $z\varphi(z)$, for the second one, it is $z\varphi[z\varphi_1(z)]$, and, in general, the distribution for an nth component is $G_n(z) = z\varphi[H_n(z)]$. Here the sequence H_n is defined by the recursion relation

$$H_n(z) = z\varphi_1[H_{n-1}(z)], \quad H_0(z) = z. \tag{B5}$$

Its stationary solution $H(z)$ satisfies the equation

$$H(z) = z\varphi_1[H(z)]. \tag{B6}$$

So $G(z) = z\varphi[H(z)] = \langle z^{M_i} \rangle$ is the transformed probability function that a randomly chosen vertex is in a connected component of size M_i.

The function $G(z)$ allows one to find, in particular, the relative size of a giant connected component, $m_\infty = N_\infty / N$. Let us consider the solutions of Eq. (B6) as $z \to +0$, $H(+0) = t_c$; $t_c = \varphi_1(t_c)$. Beside the trivial solution equal to zero, there is another solution, $t_c < 1$ (see Fig. 4),

$$t_c = \varphi_1(t_c), \quad 0 \le t_c < 1 \quad \text{if} \quad z_1 = \varphi_1'(1) = \frac{\varphi''(1)}{\varphi'(1)} > 1. \tag{B7}$$

So $\varphi(t_c)$ is the total relative size of all connected components of the network, and the relative size of the giant connected component is

$$m_\infty = 1 - \varphi(t_c). \tag{B8}$$

Note that the condition (B7) may be written as

$$\sum_q q(q-2)\Pi(q) > 0.$$

If there are no "dead ends" in the network, then $t_c = \varphi(t_c) = 0$, and almost all vertices in the network are in the giant connected component.

APPENDIX C: EQUATION FOR THE DISTRIBUTION OF τ AND AUTOCORRELATOR

Here we show how to derive (i) the key equation of this paper, namely Eq. (19) for $T(s,x)$ (which is the Laplace representation of the distribution of $\tau_{n,ij}$), (ii) its equivalent form (C7), and (iii) expression (20) for the autocorrelator in terms of $T(s,x)$.

If $n > 1$, Eq. (14) may be written as (see Fig. 3)

$$sP_{n+1,i}(s) - [P_{n,j}(s) - P_{n+1,i}(s)] + \sum_{k=1}^{b_{n+1,i}} [P_{n+1,i}(s) - P_{n+2,k}(s)]$$

$$= 0. \tag{C1}$$

Dividing both parts of the equation by $P_{n+1,i}(s)$, and taking into account the definition (17), we obtain

$$\frac{\tau_{n,ij}(s)}{1 - \tau_{n,ij}(s)} = s + \sum_{k=1}^{b_{n+1,i}} \tau_{n+1,jk}(s). \tag{C2}$$

If $n=0$, Eq. (14) takes the form

$$sP_0(s) + \sum_{k=1}^{q_0} [P_0(s) - P_{1k}(s)] = 1. \tag{C3}$$

Dividing both sides of Eq. (C3) by P_0, and taking into account Eq. (17), we obtain

$$P_0(s) = (s + \sum_{k=1}^{q_0} \tau_{0,0k})^{-1}. \tag{C4}$$

Recursive relations (C2) express the set of random variables $\tau_{n,ij}$, $n \geq 0$, in terms of the set of independent and statistically equivalent random variables $q_{m,i}$, $m > n$. It is important that the variable $\tau_{n,ij}$ depends only on the degrees of vertices belonging to the tree branch, which grows from the edge $(n,i)-(n+1,j)$. So in Eq. (C2), $\tau_{n,ij}(s)$ is expressed through $q_{n+1,j}$ independent random variables: The branching number $b_{n+1,j}=q_{n+1,j}-1$ and $b_{n+1,j}$ statistically equivalent variables $\tau_{n+1,jk}$, $k=1, \dots, b_{n+1,j}$. In the thermodynamic limit, the statistical properties of branches, starting at any distance from the initial vertex, are the same. Consequently, all random variables $\tau_{n,ij}$ are distributed equally, independently of i,j and n. Then, omitting unnecessary indices, one can rewrite Eq. (C2) as

$$e^y \exp\left(-\frac{y}{1 - \tau(s)}\right) = e^{-sy} \prod_{k=1}^{b} e^{-y\tau_k(s)}. \tag{C5}$$

The next step is averaging both the parts of Eq. (C5). We use definition (18), properties of statistical equivalence of τ, and mutual independence of the branching number b and all τ_k. We also use the following integral identity:

$$e^{-y/\alpha} = \frac{\sqrt{y}}{i\pi} \int_{-i\infty+\delta}^{+i\infty+\delta} \frac{dx}{\sqrt{x}} K_1(2\sqrt{xy}) e^{\alpha x}, \tag{C6}$$

where K_1 is the MacDonald function of index 1. Then

$$\exp\left(-\frac{y}{1 - \tau}\right) = \frac{\sqrt{y}}{i\pi} \int_{-i\infty+\delta}^{+i\infty+\delta} \frac{dx}{\sqrt{x}} K_1(2\sqrt{xy}) e^{(1-\tau)x}.$$

Finally, we have

$$e^y \frac{\sqrt{y}}{i\pi} \int_{-i\infty+\delta}^{+i\infty+\delta} \frac{dx}{\sqrt{x}} K_1(2\sqrt{xy}) e^x T(s,x)$$

$$= e^{-sy} \langle \{[T(s,y)]^b\} \rangle = e^{-sy} \varphi_1[T(s,y)], \tag{C7}$$

where definition (B4) was used. (Here b is a branching coefficient of some edge.)

Let us now derive Eq. (19). Equation (C2) can be also written as

$$\tau(s) = 1 - \frac{1}{1 + \xi(s)}, \quad \xi(s) = s + \sum_{k=1}^{b} \tau_k(s), \tag{C8}$$

where some of the indices are omitted for the sake of brevity. Now one can use the following integral identity, equivalent to Eq. (C6):

$$e^{x/\alpha} = 1 + \sqrt{x} \int_0^\infty \frac{dy}{\sqrt{y}} I_1(2\sqrt{xy}) e^{-\alpha y}, \tag{C9}$$

where I_1 is the modified Bessel function of index 1. Then

$$\langle e^{-x\tau(s)} \rangle = T(s,x) = e^{-x} \langle e^{x/[1+\xi(s)]} \rangle$$

$$= e^{-x}\left(1 + \sqrt{x} \int_0^\infty \frac{dy}{\sqrt{y}} I_1(2\sqrt{xy}) e^{-y} \langle e^{-\xi(x)} \rangle\right)$$

$$= e^{-x}\left(1 + \sqrt{x} \int_0^\infty \frac{dy}{\sqrt{y}} I_1(2\sqrt{xy}) e^{-(1+s)y} \langle [T(s,y)]^b \rangle\right)$$

$$= e^{-x}\left(1 + \sqrt{x} \int_0^\infty \frac{dy}{\sqrt{y}} I_1(2\sqrt{xy}) e^{-(1+s)y} \varphi_1[T(s,y)]\right),$$

which gives Eq. (19).

After averaging, Eq. (C4) takes the form

$$\bar{P}_0(s) = \int_0^\infty dx e^{-sx} \left\langle \prod_{k=1}^{q_0} \exp(-x\tau_{0,0k}) \right\rangle.$$

Taking into account the property of statistical independence and equivalence indicated above, we obtain Eq. (20).

APPENDIX D: DERIVATION OF THE RECURSION RELATION

Let us derive the recursion relation (23) for $U_\ell(s,y)$. Consider the following expression:

$$\frac{S_{n,ij}^{(l)}(s)}{1 - \tau_{n,ij}(s)} \exp\left(-\frac{y\tau_{n,ij}(s)}{1 - \tau_{n,ij}(s)}\right). \tag{D1}$$

Differentiating the identity (C6) with respect to x we have

$$\frac{1}{\alpha} e^{-y/\alpha} = \int_{-i\infty+\delta}^{+i\infty+\delta} \frac{dx}{\pi i} K_0(2\sqrt{xy}) \alpha e^{\alpha x}. \tag{D2}$$

Substituting $\alpha = 1/(1-\tau)$ and using the definition of U_l, Eq. (22), we transform the expression (D1) into

$$e^y \int_{-i\infty+\delta}^{+i\infty+\delta} \frac{dx}{\pi i} K_0(2\sqrt{xy}) S_{n,ij} \exp\{x[1 - \tau_{n,ij}(s)]\}, \quad \text{(D3)}$$

where K_0 is MacDonald's function of index 0. In the infinite network the expression (D3) is independent of the chosen edge $(n,i)-(n+1,j)$ and depends only on l and on s. Averaging Eq. (D3) we obtain

$$e^y \int_{-i\infty+\delta}^{+i\infty+\delta} \frac{dx}{\pi i} K_0(2\sqrt{xy}) e^x U_l(s,x). \quad \text{(D4)}$$

On the other hand, due to the treelike structure, the first multiplier in the angular brackets in Eq. (D1) may be expressed as a sum of terms with $l \to l-1$ (see Fig. 3),

$$S_{n;i,j}^{(l)}(s) = [1 - \tau_{n,ij}(s)] \sum_{k=1}^{b_{n+1,j}} S_{n+1,jk}^{(l-1)}(s). \quad \text{(D5)}$$

Using Eq. (D5) together with Eq. (C2), we see that the expression (D1) is equal to the following:

$$e^{-sy} \prod_{k=1}^{b_{n+1,i}} \exp[-y\tau_{n+1,jk}(s)] \sum_{m=1}^{b_{n+1,j}} S_{n+1,jm}^{(l-1)}(s). \quad \text{(D6)}$$

Let us average Eq. (D6) taking into account the statistical properties of the variables τ and b (or q), indicated above. Note that in each of the $b_{n+1,j}$ terms we have $b_{n+1,j}-1$ multipliers $\langle \exp[-y\tau_{n+1,jk}(s)]\rangle = T(s,y)$ with $k \neq m$, and the multiplier $\langle S_{n+1,jm}^{(l-1)}(s)\exp[-y\tau_{n+1,jk}(s)]\rangle = U_{l-1}(s,y)$. So the remaining average over $b_{n+1,j} \equiv b$ can be easily performed, which gives

$$e^{-sy} \sum_b b[T(s,y)]^{b-1}\Pi_1(b) U_{l-1}(s,y) = e^{-sy}\varphi_1'[T(s,y)]U_{l-1}(s,y). \quad \text{(D7)}$$

[Recall that the distribution function of b is $\Pi_1(b)$, Eq. (15), i.e., $\varphi_1(z)$ in Z-representation, Eq. (B4).] Equating expression (D1) to Eq. (D7), we derive the recursion relation for U_l in the following form:

$$e^y \int_{-i\infty+\delta}^{+i\infty+\delta} \frac{dx}{\pi i} K_0(2\sqrt{xy}) e^x U_l(s,x) = e^{-sy}\varphi_1'[T(s,y)]U_{l-1}(s,y). \quad \text{(D8)}$$

Equation (27) for eigenfunctions $\psi_m(s)$ can also be written as

$$e^y \int_{-i\infty+\delta}^{+i\infty+\delta} \frac{dx}{\pi i} K_0(2\sqrt{xy}) e^x \psi_m(s,x) = e^{-sy}\varphi_1'[T(s,y)]\psi_m(s,y). \quad \text{(D9)}$$

Let us now replace τ in the definition of U_l, Eq. (22), with its expression in terms of the random variable $\xi(s)$, $\tau(s) = \xi(s)/[1+\xi(s)]$. In turn, for $\xi(s)$ we use relation (C8). Again, use Eq. (D5) for S. Differentiating integral identity (C9) with respect to x gives

$$e^{x/\alpha} = \int_0^\infty dy I_0(2\sqrt{xy})\alpha e^{-\alpha y}. \quad \text{(D10)}$$

Then we have

$$U_l(s,x) = e^{-x}\int_0^\infty dy I_0(2\sqrt{xy})e^{-(1+s)y}$$

$$\times \left\langle \prod_{k=1}^{b_{n+1,i}} e^{-y\tau_{n+1,jk}(s)} \prod_{m=1}^{b_{n+1,j}} S_{n+1,jm}^{(l-1)} \right\rangle$$

$$= e^{-x}\int_0^\infty dy I_0(2\sqrt{xy})e^{-(1+s)y}$$

$$\times \sum_b b[T(s,y)]^b \Pi_1(b) U_{l-1}(s,y). \quad \text{(D11)}$$

Using Eq. (D7) for averaging over b readily leads to Eq. (23). The initial condition (24) follows directly from the definition of U_l, Eq. (22).

APPENDIX E: ASYMPTOTIC SOLUTIONS OF INTEGRAL EQUATIONS

Here we obtain the asymptotic solutions of integral equation (C7) for $T(s,x)$ in all three situations: (i) $q_m > 2$, (ii) $q_m = 2$, and (iii) $q_m = 1$.

Calculating the asymptotics at large x we replace the MacDonald functions $K_\nu(2\sqrt{xy})$ with the leading term of its asymptotic expression

$$K_\nu(z) \to \sqrt{\frac{\pi}{2z}}e^{-z}. \quad \text{(E1)}$$

This asymptotics is independent of ν. Then Eqs. (C7) and (D8) at large x take the forms

$$e^y \frac{y^{1/4}}{i\sqrt{2\pi}}\int_{-i\infty+\delta}^{+i\infty+\delta} \frac{dx}{x^{3/4}}\exp(x - 2\sqrt{xy})T(s,x) = e^{-sy}\varphi_1[T(s,y)]$$

$$\text{(E2)}$$

and

$$e^y \frac{1}{iy^{1/4}\sqrt{2\pi}}\int_{-i\infty+\delta}^{+i\infty+\delta} \frac{dx}{x^{1/4}}\exp(x - 2\sqrt{xy})U_l(s,x)$$

$$= e^{-sy}\varphi_1'[T(s,y)]U_{l-1}(s,y). \quad \text{(E3)}$$

Equation (D9) in the asymptotic limit has the form

$$e^y \frac{1}{iy^{1/4}\sqrt{2\pi}}\int_{-i\infty+\delta}^{+i\infty+\delta} \frac{dx}{x^{1/4}}\exp(x - 2\sqrt{xy})\psi_m(s,x)$$

$$= \mu_m(s)e^{-sy}\varphi_1'[T(s,y)]\psi_m(s,y). \quad \text{(E4)}$$

Let us first consider Eq. (E2) in the case $s>0$, $y \to \infty$. According to the definition (18) of $T(s,x)$, this function is the Laplace transform of the probability density of a random variable $\tau(s)$. This variable satisfies the condition $0 < \tau < 1$, as it follows e.g., from the recursion relation (C2). Hence, (i) the function $T(s,x)$ is analytic everywhere in the complex

plane x, and $T(s,x) \to 0$ as $\mathrm{Re}\, x \to \infty$; (ii) $T(s,x)$ cannot decrease with x faster than exponentially. Then $T(s,x)$ can be represented as

$$T(s,x) = A \exp[-\tau_m(s)x - \vartheta_0(s,x)], \qquad (E5)$$

where $\tau_m \geq 0$, and $\partial_x \vartheta_0(s,x) \to 0$ as $x \to +\infty$. If $q_m > 1$ (we will consider the other case separately), then φ_1 on the right-hand side of Eq. (E2) can be replaced with its leading term, $\varphi_1(z) \to q_m \Pi(q_m) z^{q_m - 1}$,

$$\frac{y^{1/4}}{2i\pi^{1/2}} \int_{-i\infty}^{+i\infty} \frac{dx}{x^{3/4}} \exp[(1 - \tau_m)x - 2\sqrt{xy} - \vartheta_0(x)]$$

$$= \frac{q_m}{\bar{q}} \Pi(q_m) A^{q_m - 2} \exp[-(1 + s)y - (q_m - 1)\tau_m y$$

$$- (q_m - 1)\vartheta_0(y)]. \qquad (E6)$$

The integral on the left-hand side may be treated in the saddle point approximation. The saddle point equation is the condition that the derivative of the function in the exponential becomes equal to zero, namely,

$$y = x[1 - \tau_m - \vartheta_0'(x)]^2. \qquad (E7)$$

This equation also expresses y in terms of x. On the right-hand side, we assume that

$$\vartheta_0(y) \approx \vartheta_0[(1 - \tau_m)^2 x] - 2(1 - \tau_m)x\vartheta_0'(x)\vartheta_0'[(1 - \tau_m)^2 x].$$

One must prove afterwards that the neglected terms of the order of $x\vartheta_0'^3$, and ones with higher derivatives, are small. If we set $\vartheta_0' = 0$ in the preexponential factor of the saddle point approximation, it reduces to 1. So we arrive at the following equation for ϑ_0:

$$\ln\left(\frac{q_m}{\bar{q}} \Pi(q_m) A^{q_m - 2}\right) + (1 - \tau_m)^2\left(s + (q_m - 1)\tau_m - \frac{\tau_m}{1 - \tau_m}\right)x$$

$$- 2(1 - \tau_m)\left(s + (q_m - 1)\tau_m - \frac{\tau_m}{1 - \tau_m}\right)x\vartheta_0'(x)$$

$$+ (q_m - 1)\vartheta_0[(1 - \tau_m)^2 x] - \vartheta_0(x) + [1 + s$$

$$+ (q_m - 1)\tau_m]x\vartheta_0'^2(x) - 2(q_m - 1)(1 - \tau_m)x\vartheta_0'(x)$$

$$\times \vartheta_0'[(1 - \tau_m)^2 x] = 0. \qquad (E8)$$

The main term of this equation, linear in x, reduces to zero if

$$\frac{\tau_m}{1 - \tau_m} = s + (q_m - 1)\tau_m. \qquad (E9)$$

It also reduces the third term in Eq. (E8) to zero. Suppose that our network is a regular Bethe lattice with the coordination number q_m. Then Eq. (C7), equivalent to Eq. (20), has the exact solution $T(s,x) = \exp[-\tau_m(s)x]$, where $\tau_m(s)$ is the proper solution of Eq. (E9). This $\tau_m(s)$ is a regular function of s as $s > s_c = -q_m + 2\sqrt{q_m - 1}$, $s_c < 0$, and $\tau_m(s_c) \equiv \tau_c = 1 - 1/\sqrt{q_m - 1}$. At $s = s_c$, $\tau_m(s)$ has a square root singularity. So in the regular Bethe lattice, the density of Laplacian eigenvalues $\rho(\lambda)$ is nonzero at $\lambda > \lambda_c = -s_c = q_m - 2\sqrt{q_m - 1}$, and $\rho(\lambda) \sim \sqrt{\lambda - \lambda_c}$ at $\lambda - \lambda_c \ll 1$. Thus, we can conclude, that for any network with $q_m > 1$, the edge of the spectrum is λ_c

≥ 0. Moreover, $\lambda_c(q_m > 2) > 0$. In random networks, the asymptotics of $\rho(\lambda)$ turn out to be sharply different from a regular Bethe lattice.

Requiring that the main correction to the leading term in Eq. (E8) also asymptotically vanish gives $(q_m - 1)\vartheta_0[(1 - \tau_m)^2 x] = \vartheta_0(x)$. This equality is satisfied when

$$\vartheta_0(x) = Bx^\alpha, \quad \alpha = \frac{\ln(q_m - 1)}{2 \ln[1/(1 - \tau_m)]} = \frac{\ln[1/(1 - \tau_c)]}{\ln[1/(1 - \tau_m)]}. \qquad (E10)$$

If $s > s_c$, then $\tau_m > \tau_c$ and $\alpha < 1$. This means that all approximations made during the derivation are justified. Therefore, the integral in Eq. (20) is convergent, and $\bar{P}_0(s)$ is a regular function of s. If, however, $s \to s_c$, then $\alpha \to 1$. That is, the last two terms in Eq. (E8) should be also taken into account when s is close to s_c. In this region of s, Eq. (E8) and a similar equation for the asymptotics of $\psi_0(x)$, which can be derived from Eq. (D9), must be treated in different ways for $q_m > 2$ and for $q_m = 2$. Note that if $q_m = 1$, then Eq. (E8) must be replaced with a slightly different equation.

1. Minimum degree $q_m > 2$

In this case we can set the first term in Eq. (E8) to 0, properly choosing the value of the constant A in Eq. (E5). We set $T(s,x) = \exp[-\vartheta(\delta,x)]$. Here $\vartheta(\delta,x) = \vartheta_0 + \tau_m x$ includes, besides ϑ_0, also a slowly varying linear term $\tau_m x$. Here we introduce a small variable $\delta = \sqrt{s - s_c}$.

First, let us consider Eq. (E8) at $s = s_c$ and $\tau = \tau_c$. We have

$$(q_m - 1)\vartheta\left(\frac{x}{q_m - 1}\right) - \vartheta(x) + \sqrt{q_m - 1}\, y\vartheta'^2(x)$$

$$- 2\sqrt{q_m - 1}\, y\vartheta'(x)\vartheta'\left(\frac{x}{q_m - 1}\right) = 0.$$

Now we make the substitution $\vartheta(x) = (q_m - 1)^{-1/2} x\chi(\ln x)$. We assume that χ is a small and slowly varying function of its argument. Then, we make the following approximations, which must be justified afterwards. Replace $\chi[z - \ln(q_m - 1)]$ in the first term with $\chi(z) - \chi'(z)\ln(q_m - 1)$, where $z = \ln y$, and neglect all derivatives of χ in the last two terms. As a result we obtain

$$\ln(q_m - 1)\chi'(z) + \chi^2(z) = 0.$$

This equation has the solution $\chi(z) = \ln(q_m - 1)/(z + c)$, where $c \sim 1$ is some constant of integration. Thus, finally, we obtain

$$\vartheta(\delta = 0, x) = \frac{x \ln(q_m - 1)}{\sqrt{q_m - 1} \ln(Cx)}. \qquad (E11)$$

Now assume $|\delta|^2 = |s - s_c = |\lambda - \lambda_c|| \ll 1$. The first term in Eq. (E8) reduces to $\delta^2 y / (q_m - 1)$. We neglect the second term of the equation, assuming it to be small. After the same set of substitutions and approximations as in the case $s = s_c$, we have the following equation for $\chi(z)$:

$$\ln(q_m - 1)\chi'(z) + \chi^2(z) = \frac{\delta^2}{\sqrt{q_m - 1}}.$$

Solving this equation, we obtain the following result for $\vartheta(\delta, x)$:

$$\vartheta(\delta, x) = \frac{\delta x}{(q_m - 1)^{3/4}}\coth\left(\frac{\delta \ln(Cx)}{(q_m - 1)^{1/4}\ln(q_m - 1)}\right).$$
(E12)

After substitution $\delta = \sqrt{s - s_c}$, this turns into Eq. (51).

2. Minimum degree $q_m = 2$

Here $\tau_m(s) \to 0$ as $s \to 0$. Then at small $|s|$ we can consider $\vartheta(s, x) = \vartheta_0(s, x) + \tau_m(s)x$ as a slowly varying function. When calculating the integral in Eq. (E2) in the saddle point approximation, we take also into account the preexponential factor as a correction, though it is close to 1. Replacing on the right-hand side $\varphi_1(T)$ with its leading term, linear on T, and taking into account the saddle point equation $y = x[1 - \vartheta'(x)]^2$, we have

$$\left(\frac{1 - \vartheta'(x)}{1 - \vartheta'(x) - 2x\vartheta''(x)}\right)^{1/2}\exp[x\vartheta'^2(x) - \vartheta(x)]$$

$$= \frac{2\Pi(2)}{\bar{q}}\exp[-sx - \vartheta(x) + 2x\vartheta'^2(x)].$$
(E13)

Here we omitted negligibly small terms, $2sx\vartheta'(x)$ and others.

Accounting for the smallness of ϑ' and $x\vartheta''$, we obtain the equation

$$x\vartheta'^2(x) - y\vartheta''(x) = sx + \frac{a}{\pi}, \quad a = \pi \ln\left(\frac{\bar{q}}{2\Pi(2)}\right) > 0.$$
(E14)

We assume that the first term on the left-hand side is small and search for the solution of this equation in the form $\vartheta = \vartheta_1 + \vartheta_2$. Here ϑ_1 must be found from $x\vartheta_1'^2 = sx + a/\pi$. At $s = 0$ we find $\vartheta_1 = 2\sqrt{ax/\pi} + c$, where C is some constant of integration, $C \sim 1$. At $s \neq 0$, performing the integration, we have

$$\vartheta_1(x) = \frac{1}{\sqrt{s}}f(sx) + C, \quad f(z) = \sqrt{z(a/\pi + z)} + \frac{a}{\pi}\text{arcsinh}\sqrt{\frac{\pi z}{a}}.$$
(E15)

In principle, here $C = C(s)$, but for small s one can set $C(s) = C = C(0)$. For ϑ_2 we have $2\vartheta_1'\vartheta_2' = \vartheta_1''$. Therefore, up to the constant, $\vartheta_2 = (\ln \vartheta_1')/2$. As a result, we have asymptotically the expression (63) for $\vartheta = \vartheta_1 + \vartheta_2$.

We replace in Eq. (D9) K_0 with its asymptotic (E1) at large values of argument. Then, taking into account that $T(s, x)$ is small at large x, we replace $\varphi_1'(T)$ on the right-hand side with its value of zero argument, $2\Pi(2)/\bar{q}$. As a result, we arrive at the following equation for $\psi_0(s, x) = \exp[-\varkappa(s, x)]$:

$$\frac{e^y}{2iy^{1/4}\sqrt{\pi}}\int_{-i\infty+\delta}^{+i\infty+\delta}\frac{dx}{x^{1/4}}\exp[-2\sqrt{xy} + x - \varkappa(x)]$$

$$= \mu_0(s)\frac{2}{\bar{q}}\Pi(2)\exp[-sy - \varkappa(y)].$$
(E16)

This equation differs from Eq. (E6) for $\vartheta(x) = \vartheta_0(x) + \tau_m(s)x$ only in the preexponential factor on the left-hand side. Quite analogously to Eq. (E14), we obtain an equation for $\varkappa$,

$$x\varkappa'^2(x) - x\varkappa''(x) - \varkappa'(x) = sy + \frac{a'}{\pi},$$

$$a' = \pi \ln\left(\frac{\mu_0(s)\bar{q}}{2\Pi(2)}\right) \approx a > 0.$$
(E17)

Comparing the above equation with Eq. (E14), we conclude that $\varkappa(x) = \vartheta_1(x) + \varkappa_2(x)$, where $\vartheta_1(x)$ is given by Eq. (E15), and $\varkappa_2$ must be found from $2x\vartheta_1'(x)\varkappa_2'(x) = x\vartheta_1''(y) + \vartheta_1'(x)$. The solution is $\varkappa_2 = (\ln x\vartheta_1')/2$. As a result, accounting for Eq. (E15), we obtain the expression (66) for $\psi_0 = \exp(-\varkappa)$.

3. Minimum degree $q_m = 1$

To obtain the equation for $\Theta(z) = \lim_{s \to 0} T(s, z/s)$, let us start with Eq. (E2), which is valid as $y \to \infty$. Let us replace y in this equation with z/s simultaneously changing the integration variable, $x = \zeta/s$. Then we have

$$\frac{z^{1/4}}{i\sqrt{2\pi s}}\int_{-i\infty+\delta}^{+i\infty+\delta}\frac{d\zeta}{\zeta^{3/4}}\exp\left(\frac{(\sqrt{z} - \sqrt{\zeta})^2}{s}\right)T\left(s, \frac{\zeta}{s}\right) = e^{-z}\varphi_1[T(s, z/s)].$$
(E18)

In the limit $s \to 0$ the saddle point approximation becomes exact, with the saddle point condition simply $\zeta_c = z$. So, assuming that the limit (35) of the function T exists, we immediately arrive at Eq. (36) for $\Theta(z)$. The recursion relation (41) is obtained in the same way by using the asymptotic equation (E3).

We can reasonably assume that at small s and large x,

$$T(s, x) = \Theta(sx) + \exp[-\vartheta(s, x)],$$
(E19)

where the last term is small. Substituting this into Eq. (E2) and linearizing the right-hand side with respect to $e^{-\vartheta}$, we obtain

$$\frac{e^y y^{1/4}}{2i\sqrt{\pi}}\int_{-i\infty+\delta}^{+i\infty+\delta}\frac{dx}{x^{3/4}}\exp[-2\sqrt{xy} + x - \vartheta(x)]$$

$$= \varphi_1'[\Theta(sy)]\exp[-sy - \vartheta(y)].$$
(E20)

Repeating the steps leading to Eq. (E14), we obtain

$$x\vartheta'^2(x) - x\vartheta''(x) = sx - \ln \varphi_1'[\Theta(sx)].$$
(E21)

Treating the second term on the left-hand side as a perturbation, we set $\vartheta = \vartheta_1 + \vartheta_2$, $|\vartheta_2| \ll |\vartheta_1|$. The equations for ϑ_1 and ϑ_2 are

$$x\vartheta_1'^2(x) = sx - \ln \varphi_1'[\Theta(sx)], \quad 2\vartheta_1'(x)\vartheta_2'(x) = \vartheta_1''(x).$$

Then one can easily obtain

$$\vartheta_1(s,x) = \frac{1}{\sqrt{s}} g(sx) + C,$$

$$g(z) = \int_0^z d\zeta \sqrt{1 - \frac{\ln \varphi_1'[\Theta(\zeta)]}{\zeta}},$$

$$\vartheta_2(s,x) = \frac{1}{4}\ln[sg'^2(sx)], \qquad (E22)$$

where $C \sim 1$. Finally, for $\vartheta(x) = \vartheta_1(x) + \vartheta_2(x)$ we obtain formula (72).

The function $g(z)$ has a singularity point at $z = z_s < 0$, when the expression in the square root under the integral becomes 0, i.e., when $\varphi_1'[\Theta(z_s)] = e^{z_s}$. Here z_s is defined as $t_s = \Theta(e^{-z_s})$, t_s is the solution of Eq. (73) (see Fig. 4). It is obvious that $g(z_s) \equiv ia$ is imaginary, because the expression in the square root in Eq. (E22) is negative. The calculation of a may be simplified if we replace the integration variable ζ with $\xi = \Theta(\zeta)$. We use the definition of the function Θ, Eq. (36), from which $\zeta = \ln[\varphi_1(\xi)/\xi]$ follows, so $d\zeta = [\varphi_1'(\xi)/\varphi_1(\xi) - 1/\xi]d\xi$. We substitute these relations into the integral for $g(z_s)$ in Eq. (E22), and take into account that $\xi = t_c = \Theta(0)$ at the lower limit of integration, $\zeta = 0$ while $\xi = t_s$ on the upper limit. This gives Eq. (75). Note that $z = t_s$ is a singularity point of the function $\Theta(z)$. Since the derivative of the reverse function $z = \ln[\varphi_1(\Theta)/\Theta]$ is zero, $\Theta(z_s + \eta) = t_s + O(\sqrt{\eta})$ at small $|\eta|$. Therefore, in Eq. (E22) $\ln \varphi_1'[\Theta(z_s + \eta)]/(z_s + \eta) - 1 \sim \sqrt{\eta}$ and so $g(z_s + \eta) = ia + O(\eta^{5/4})$.

The calculation of the asymptotics of the eigenfunction $\psi_0(s,x)$ is quite similar to that for $q_m = 2$. Let us represent $\psi_0(s,x) = \exp[-\varkappa(s,x)]$. From Eq. (D9) we obtain the integral equation for $\varkappa$ which differs from Eq. (E16) only in that the constant $\varphi_1'(0) = 2\Pi(2)/\bar{q}$ should be replaced with the function $\varphi_1'[\Theta(sy)]$. Proceeding further, we have

$$x\varkappa'^2(x) - x\varkappa''(x) - \varkappa'(x) = sy - \ln \varphi_1'[\Theta(sx)].$$

This equation for $\varkappa$ differs from Eq. (E17) only by the last term on the right-hand side. So, as at $q_m = 2$, we have asymptotically $\psi_0(s,x) \sim x^{-1/2}T(s,x) \sim x^{-1/2}\exp[-\vartheta(s,x)]$, where $\vartheta(s,x)$ is given by Eq. (72).

APPENDIX F: EIGENFUNCTION WITH MINIMUM CHARACTERISTIC NUMBER AT $s = 0$

Here we show that when $s \to 0$, the lowest eigenvalue μ_0 and the corresponding eigenfunction $\psi_0(x)$ of Eq. (27) are 1 and $-T_0'(x)$, respectively.

At $s = 0$, Eq. (19) takes the form

$$T_0(x) = e^{-x}\left(1 + \sqrt{x}\int_0^\infty \frac{dy}{\sqrt{y}}I_1(2\sqrt{xy})e^{-x}\varphi_1[T_0(y)]\right). \quad (F1)$$

Let us differentiate both the parts of this relation with respect to x. It is easy to check the identity

$$\frac{\partial}{\partial x}\left(e^{-x-y}\sqrt{\frac{x}{y}}I_1(2\sqrt{xy})\right) = -\frac{\partial}{\partial y}[e^{-x-y}I_0(2\sqrt{xy})].$$

After differentiating and using this identity, we integrate by parts on the right-hand side. The integrated term e^{-x} on the lower limit of integration, $y = 0$, will be cancelled by the result of differentiating the e^{-x}. Finally, we have

$$T_0'(x) = e^{-x}\int_0^\infty dy I_0(2\sqrt{xy})e^{-y}\varphi_1'[T_0(y)]T_0'(y). \quad (F2)$$

Comparing this with Eq. (27) at $s = 0$, we see that $-T_0'(x)$ is the (unnormalized) eigenfunction of this equation corresponding to the eigenvalue $\mu_0 = 1$. It is known from the theory of linear integral equations [31] that the eigenfunction corresponding to the maximum characteristic number can be chosen to be real and positive within the interval of integration. One can see that $-T_0'(x) > 0$ at any $x > 0$, and the corresponding eigenvalue $\mu_0 = 1$ is indeed a maximal one.

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[32] In fact, we applied a slightly different procedure. After step (i) we knew only the asymptotic of $T(s,x)$ at $\mathrm{Re}\,x \gg 1$ and $|s - s_c| \ll 1$. The analytic continuation of $T(s,x)$ as a function of the variable s to the area $s < s_c$ has a singularity in terms of the variable x at large x. Therefore, to calculate $\rho(\lambda)$ at $\lambda > \lambda_c$, one can use (50). Then the asymptotics of $\rho(\lambda)$ at $\lambda \to \lambda_c + 0$ is defined by the position of the singularity with respect to the variable x, and by the behavior of $T(-\lambda,x)$ near this singularity.

[33] Alternatively, find eigenfunctions and eigenvalues, $\psi_m(s,x)$ and $\mu_m(s)$, of the integral equation (27). Then find $\bar{P}_l(s)$ by using Eqs. (31)–(33). Here the nontrivial part of the propagator, describing its relaxation to a stationary value, corresponds to the discrete part of the spectrum of characteristic numbers of this equation.

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